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The theory of overdetermined linear systems and its applications to non-linear field equations

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Abstract. *We review some aspects of the theory of overdetermined linear systems of partial differential equations and use it to interpret some non-linear equations of classical field theory as integrability conditions of linear one. In particular, it is shown that the Einstein and the Yang-Mills equations are equivalent to the existence of flat connections in affine subspaces of connections on some vector bundles, i.e. they may be written as zero-curvature conditions.*

1. INTRODUCTION

The interpretation of some non-linear field theoretic models [1, 2] in dimension 2 as integrability conditions of linear systems and, more precisely, as zero curvature conditions for family of connections has been very useful for the analysis of these models. On the other hand some more realistic models such as those described by Yang-Mills or Einstein equations do present themselves as the vanishing of certain covariant part of curvatures, it is therefore natural to try to interpret these models also as integrability conditions for some linear systems and one may even wonder whether they can be represented in some sense as zero curvature conditions. We shall here report on some recent work on this subject [3, 4, 5, 6]. There is another even more natural reason for physicists to study integrability conditions; namely it is well known that there are

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troubles with some linear classical field equations in external fields [7]. These troubles are connected with the non-integrability of these equations for generic external fields; there are however some configurations of these external fields for which the systems are integrable. It is therefore useful to have an algebraic way to produce the obstructions to integrability of systems of partial differential equations. This is one of the goal of the formal (or analytic) theory of overdetermined systems of partial differential equations [8] developed by D.C. Spencer [9], D.G. Quillen [10], H. Goldschmidt [11, 12], B. Malgrange [13, 14] and some other ones. In view of the applications we have in mind we shall only describe here some aspects of that theory for linear systems, the theory for non-linear systems [12] is basically similar but with some technical complications.

They key notion is the notion of formal integrability. This is a very natural notion: one tries to solve a system of partial differential equations at order ℓ at some point (i.e. in the sense of Taylor expansion at order ℓ at that point) and, roughly speaking, one says that such system is formally integrable whenever there are no obstructions to continue the expansion from order ℓ to order $\ell + 1$, for any ℓ and any point. If the system is a good scalar equation or more generally if one can solve the local Cauchy problem for it, then everything is O.K. and one does not learn so much with such expansions. In some sense however the converse also works at the level of analytic equations and solutions; i.e. roughly speaking, if the system is formally integrable the local Cauchy problem, for non characteristic data, is soluble (at the analytic level) [15].

The obstructions to formal integrability of a system essentially take their values in some space of cohomology constructed with the symbol of the system.

The appropriate language to deal with Taylor expansion in a compact coordinate-free way is the language of jets [16]; it is why we start with a section (section 2) dealing with differential operators and jet bundles. In section 3, we define linear equations and their prolongations in terms of jet bundles and start to discuss the notion of formal integrability. In section 4 we describe the relevant cohomology of symbol in terms of finite dimensional vector spaces, some examples of useful symbols are described there. Section 5 gives the relevant integrability criteria and description of the obstructions to integrability. Some applications to classical field theory connected with the motivations given at the beginning of this introduction are described in section 6; it is shown, in particular, that Yang-Mills and Einstein equations may be written as zero curvature conditions (at the analytic level).

When there are indices; we use the Einstein convention of summing repeated up-down indices.

2. DIFFERENTIAL OPERATORS AND JET BUNDLES

2.1. DEFINITIONS. Let E be a smooth vector bundle over B; the space $\Gamma(E)$ of all smooth sections of E is naturally a module over the algebra $C^\infty(B)$ of all smooth functions on B. Let F be another smooth vector bundle over B and let us denote by $\mathscr{D}_0(E, F)$ the space of all homomorphisms of $C^\infty(B)$-modules from $\Gamma(E)$ in $\Gamma(B)$. It is well known that for any $L \in \mathscr{D}_0(E, F)$ there is a unique vector bundle homomorphism $p_0(L) : E \to F$ for which we have $Ls = p_0(L) \circ s$, for any $s \in \Gamma(E)$, and that, $L \mapsto p_0(L)$ allows to identify $\mathscr{D}_0(E, F)$ with the space of all vector bundle homomorphisms of E in F. Let $\mathscr{L}(E, F)$ be the space of all mappings of $\Gamma(E)$ in $\Gamma(F)$ which are linear for the underlying vector space structures (multiplication by constant functions on B); one may identify $\mathscr{D}_0(E, F)$ in $\mathscr{L}(E, F)$ by

$$\mathscr{D}_0(E, F) = \{L \in \mathscr{L}(E, F) \mid L \circ f - f \circ L = 0,\ \forall f \in C^\infty(B)\}$$

where $f \in C^\infty(B)$ is identified with the element of $\mathscr{D}_0(E, E)$ (resp. $\mathscr{D}_0(F, F)$), $s \mapsto f \cdot s$, $s \in \Gamma(E)$, (resp. $s \in \Gamma(F)$). One defines inductively, for any integer $k \geqslant 1$, the spaces $\mathscr{D}_k(E, F)$ by

$$\mathscr{D}_k(E, F) = \{L \in \mathscr{L}(E, F) \mid L \circ f - f \circ L \in \mathscr{D}_{k-1}(E, F)\}.$$

The elements of $\mathscr{D}_k(E, F)$ are called *k-th order differential operators from E in F*.

If $L \in \mathscr{D}_k(E, F)$ and $L' \in \mathscr{D}_\ell(F, G)$ then one verifies that the composition $L' \circ L = L'L$ is in $\mathscr{D}_{k+\ell}(E, G)$. One also verifies, by induction on k, that if $L \in \mathscr{D}_k(E, F)$ and if $s \in \Gamma(E)$ vanishes on some open set $\mathscr{O} \subset B$ then $Ls \in \Gamma(F)$ also vanishes on $\mathscr{O}$; thus the germ of Ls at $b \in B$ does only depend on the germ of s at $b \in B$.

Let s and s' be in $\Gamma(E)$ and $b \in B$; we say that s and s' agree to order k at b if their components in some local trivialisation have the same derivatives of order $\leqslant k$ with respect to some coordinates system at b. This is an equivalence relation which does not depend on the local trivialisation and on the coordinates: the quotient $J_{b,k}(E)$ is the set of *k-jets of sections of E* at b and we denote by $j_b^k : \Gamma(E) \to J_{b,k}(E)$ the canonical projection. $\bigcup_{b \in B} J_{b,k}(E) = J_k(E)$ is a vector bundle over B in a natural manner and $j^k : \Gamma(E) \to \Gamma(J_k(E))$ is a k-th order differential operator; $j^k \in \mathscr{D}_k(E, J_k(E))$. $J_k(E)$ is *the bundle of k-jets of sections of E*. The pair $(j^k, J_k(E))$ is characterized (up to an isomorphism of the appropriate category) be the following property.

2.2. PROPOSITION. *(Universal property of (j^k, J_k)). For any $L \in \mathscr{D}_k(E, F)$, there is a unique vector bundle homomorphism from $J_k(E)$ in F,*

$p_k(L) \in \mathscr{L}_0(J_k(E), F)$ *such that we have:* $L = p_k(L) \circ j^k$.

Thus, just as, for vector spaces, the canonical bilinear map of $E_1 \times E_2$ in $E_1 \otimes E_2$ allows to replace bilinear maps from $E_1 \times E_2$ in F by linear maps from $E_1 \otimes E_2$ in F, the pair $(j^k, J_k(E))$ allows, for vector bundles, to replace k-th order differential operators from E in F by vector bundles homomorphisms from $J_k(E)$ in F. This proposition is of course an easy consequence of the definitions; let us give some examples of applications.

2.3. Examples

1. *Functoriality of* J_k. Let α be a vector bundle homomorphism of E in F, i.e. $\alpha \in \mathscr{L}_0(E, F)$; then $j^k \circ \alpha$ is in $\mathscr{D}_k(E, J_k(F))$ and $p_k(j^k \circ \alpha)$, which will be denoted by $J_k(\alpha)$ is a vector bundle homomorphism of $J_k(E)$ in $J_k(F)$. One verifies that, if β is a vector bundle homomorphism of F in G, we have: $J_k(\beta \circ \alpha) = J_k(\beta) \circ$ $\circ J_k(\alpha)$.

2. *The canonical projections* $\pi^k_\ell : J_k(E) \to J_\ell(E)$, $(k \geqslant \ell)$.
If $k \geqslant \ell$, $j^\ell \in \mathscr{D}_\ell(E, J_\ell(E))$ is also in $\mathscr{D}_k(E, J_\ell(E))$, (since $\mathscr{D}_k(E, F) \supset \mathscr{D}_\ell(E, F)$ for $k \geqslant \ell$).
Then, $p_k(j^\ell)$ which will be denoted by π^k_ℓ, is nothing but *the canonical projection* of $J_k(E)$ on $J_\ell(E)$ obtained by «forgetting the derivatives of order strictly greater than ℓ».

3. *Prolongations.* Let L be in $\mathscr{D}_k(E, F)$. Then $j^\ell \circ L$ is in $\mathscr{D}_{k+\ell}(E, J_\ell(F))$; it is called *the ℓ-th prolongation of L*. $p_{k+\ell}(j^\ell \circ L)$ is a vector bundle homomorphism of $J_{k+\ell}(E)$ in $J_\ell(F)$ which we denote by $pr^{(\ell)}(p_k(L))$ and call *the ℓ-th prolongation of $p_k(L)$*. $pr^{(\ell)}(\phi)$ is clearly defined for any vector bundle homomorphism of $J_k(E)$ in F, (take $L = \phi \circ j^k$). Notice that, for $k = 0$, $pr^{(\ell)}(\alpha) =$ $= J_\ell(\alpha)$ corresponds to the first example.

4. *The canonical inclusions* $J_{k+\ell}(E) \subset J_\ell(J_k(E))$. If we take $L = j^k$ $(F = J_k(E))$ in the previous example, we obtain an injective vector bundle homomorphism $p_{k+\ell}(j^\ell \circ j^k)$ of $J_{k+\ell}(E)$ in $J_\ell(J_k(E))$ which allows to make the identification $J_{k+\ell}(E) \subset J_\ell(J_k(E))$ and finally to consider these bundles as subvector bundles of $(J_1)^{k+\ell}(E) = \overbrace{J_1(J_1(\dots(J_1}^{k+\ell}(E))\dots))$. With these identifications, *we have in* $(J_1)^{k+\ell}(E)$, $J_{k+\ell}(E) = J_\ell(J_k(E)) \cap J_{\ell-1}(J_{k+1}(E))$ *for any* $k \geqslant 0$ *and* $\ell \geqslant 1$ as it is easily seen by taking local coordinates, (commutativity of partial derivatives).

2.4. PROPOSITION. *Let $S^k T^*$ denote the k-th symmetric power of the cotangent* *bundle T^* of B. Then $S^k T^* \otimes E$ is canonically a sub-vector bundle of $J_k(E)$ which is the kernel of the canonical projection $\pi^k_{k-1} : J_k(E) \to J_{k-1}(E)$.*

In other words we have an exact sequence

$$0 \to S^k T^* \otimes E \xrightarrow{\epsilon} J_k(E) \xrightarrow{\pi^k_{k-1}} J_{k-1}(E) \to 0.$$

Indeed it is well known that derivative of order k at a point $b \in B$ is a well defined tensorial object whenever the derivatives of order lower than k vanish at b: $\epsilon(\mathrm{d}f_1 \vee \ldots \vee \mathrm{d}f_k \otimes s)(b) = j^k_b((f_1 - f_1(b)) \ldots (f_k - f_k(b))\, s)$, $\forall f_1, \ldots, f_k \in C^\infty(B)$, $\forall s \in \Gamma(E)$. In the following we shall make the identification $S^k T^* \otimes E \subset J_k(E)$.

2.5. DEFINITION. Let $L \in \mathscr{D}_k(E, F)$ be a k-th order differential operator. Then the restriction to $S^k T^* \otimes E$ of $p_k(L)$ is a vector bundle homomorphism $\sigma_k(L)$ of $S^k T^* \otimes E$ in F which is called *the symbol (or principal symbol) of L*. For the ℓ-th prolongation, it is easy to show that $\sigma_{k+\ell}(j^\ell \circ L)$ takes its values in $S^\ell T^* \otimes F (\subset J_\ell(F))$ and that the corresponding homomorphism $pr^{(\ell)}(\sigma_k(L))$, (which we aslo denote by $\sigma^{(\ell)}_k(L)$), of $S^{k+\ell} T^* \otimes E$ in $S^\ell T^* \otimes F$ does only depend on the homomorphism $\sigma_k(L)$ of $S^k T^* \otimes E$ in F : $pr^{(\ell)}(\sigma_k(L))$ is the composition of the canonical inclusion $S^{k+\ell} T^* \otimes E \underset{\to}{\subset} S^\ell T^* \otimes S^k T^* \otimes E$ with

$$\mathrm{Id}_{S^\ell T^*} \otimes \sigma_k(L) : S^\ell T^* \otimes (S^k T^* \otimes E) \to S^\ell T^* \otimes F.$$

$pr^{(\ell)}(\sigma_k(L))$ is called *the ℓ-th prolongation of $\sigma_k(L)$* and $pr^{(\ell)}(\varphi)$ is defined for any vector bundle homomorphism of $S^k T^* \otimes E$ in F.

Let $L \in \mathscr{D}_k(E, F)$ be a k-th order differential operator; then the kernel $\ker p_k(L) = p_k(L)^{-1}(0) \subset J_k(E)$ is not automatically a sub-bundle of $J_k(E)$ because the dimension of $\ker p_k(L) \cap J_{b,k}(E)$ may jump at some points $b \in B$ (although it is upper-semi-continuous). *L will be said to be regular* whenever $\ker p_k(L)$ is a sub-bundle of $J_k(E)$, i.e. whenever $b \mapsto \dim(\ker p_k(L) \cap J_{b,k}(E))$ is constant, (we always assume that B is connected); L will be said to be *completely regular* whenever its prolongations $j^\ell \circ L$ are regular for any $\ell \geqslant 0$.

In the following, we shall only consider regular operators; this forbids, for instance, an operator like $x^\mu \partial_\mu + V(x)$ which is not regular in any neighbourhood of the origin $x = 0$ of $\mathbb{R}^n$.

2.6. Example

Let us recall that a *connection on the vector bundle E* over B is a linear mapping, $\nabla : \Gamma(E) \to \Gamma(T^* \otimes E)$, satisfying $\nabla(f \cdot s) = \mathrm{d}f \otimes s + f \cdot \nabla s$, for any $f \in C^\infty(B)$ and $s \in \Gamma(E)$. It follows that $s \mapsto [\nabla, f]\, s = \mathrm{d}f \otimes s$ is a $C^\infty(B)$-module homomorphism and that therefore, ∇ is a 1-th order differential operator from E in $T^* \otimes E$;

$\nabla \in \mathscr{D}_1(E, T^* \otimes E)$. The corresponding vector bundle homomorphism $p_1(\nabla) : J_1(E) \to T^* \otimes E$ is such that its restriction $\sigma_1(\nabla)$ to $T^* \otimes E \subset J_1(E)$, (i.e. the symbol of ∇), is the identity mapping of $T^* \otimes E$ on itself (and this property of the symbol characterizes the connections). Another way to say the same thing consists in saying that $p_1(\nabla)$ is a *splitting* of the exact sequence

$0 \to T^* \otimes E \xrightarrow{\iota} J_1(E) \xrightarrow{\pi_0^1} E \longrightarrow 0$; thus $p_1(\nabla) \oplus \pi_0^1 : J_1(E) \to (T^* \otimes E) \oplus E$

is an isomorphism and the canonical inclusion $T^* \otimes E \subset J_1(E)$ corresponds to $T^* \otimes E \to (T^* \otimes E) \oplus \{0\} \subset (T^* \otimes E) \oplus E$. ∇ is regular but generally not completely regular. One extends ∇ to $\Gamma(\Lambda^{\cdot} T^* \otimes E)$, i.e. to E-valued differential forms on B, ($\Lambda^{\cdot} T^* = \mathbb{R} \oplus T^* \oplus \ldots \oplus \Lambda^n T^*$), by $\nabla\, \omega \otimes s = \mathrm{d}\omega \otimes s + (-1)^p \omega \wedge \nabla s$ for any p-form $\omega \in \Gamma(\Lambda^p T^*)$ and $s \in \Gamma(E)$; thus

$$\nabla : \Gamma(\Lambda^p T^* \otimes E) \to \Gamma(\Lambda^{p+1} T^* \otimes E).$$

We have $\nabla^2 f \cdot s = \nabla(\mathrm{d}f \otimes s + f \nabla s) = -\mathrm{d}f \wedge \nabla s + \mathrm{d}f \wedge \nabla s + f \nabla^2 s = f \nabla^2 s$, so $\nabla^2 : \Gamma(E) \to \Gamma(\Lambda^2 T^* \otimes E)$ is of order zero and thus of the form $s \mapsto \nabla^2 s = \Omega s$, where Ω is a 2-form with values in the bundle $\mathrm{End}(E) = E \otimes E^*$ of endomorphisms of E. We have on $\Gamma(\Lambda^{\cdot} T^* \otimes E) : \nabla^2 \psi = \Omega \wedge \psi$, for $\psi \in \Gamma(\Lambda^{\cdot} T^* \otimes E)$ (with obvious notations).

$\Omega \in \Gamma(\Lambda^2 T^* \otimes \mathrm{End}(E))$ is called *the curvature of the connection* ∇. Notice that $\nabla^2 : \Gamma(E) \to \Gamma(\Lambda^2 T^* \otimes E)$ factorizes through the 2-th order partial differential operator $j^1 \circ \nabla : \Gamma(E) \to \Gamma(J_1(T^* \otimes E))$, (i.e. the 1-th prolongation of ∇), and a vector bundle homomorphism of $J_1(T^* \otimes E)$ in $\Lambda^2 T^* \otimes E$, in spite of the fact that ∇^2 turns out to be itself a vector bundle homomorphism, (product by Ω; of course one has $\mathscr{D}_0(E, \Lambda^2 T^* \otimes E) \subset \mathscr{D}_2(E, \Lambda^2 T^* \otimes E)$).

2.7. Coordinates

Let $\mathscr{C} \subset B$ be diffeomorphic to $\mathbb{R}^n$ and let $b \to (x^1, \ldots, x^n) = (x^\mu)$ be a corresponding coordinates system on $\mathscr{C}$. Then the restriction $E \upharpoonright \mathscr{C}$ of E to $\mathscr{C}$ is a trivialisable vector bundle, i.e. it is isomorphic to $E_0 \times \mathscr{C} \simeq E_0 \times \mathbb{R}^n$ where E_0 is a finite dimensional vector space with $\dim(E_0) = \mathrm{rank}(E)$. Thus elements $\mathscr{G}$ of $E \upharpoonright \mathscr{C}$ are represented, in such trivialisation, by pairs $(\psi, x^\mu) \in E_0 \times \mathbb{R}^n$ and the bundle projection corresponds to $(\psi, x^\mu) \mapsto (x^\mu)$. Correspondingly, $J_k(E) \upharpoonright \mathscr{C}$ is isomorphic to $\left(\bigoplus_{m=0}^{m=k} S^m(\mathbb{R}^n)^* \otimes E_0\right) \times \mathbb{R}^n$ and the associated coordinates are $(\psi_{\mu_1^k \ldots \mu_k^k}, \ldots, \psi_{\mu_1^m \ldots \mu_m^m}, \ldots, \psi_{\mu_1^1}, \psi, x^\mu)$ where the $\psi_{\mu_1 \ldots \mu_m} \in E_0$ are completely symmetric in the indices $\mu_1, \ldots, \mu_m$ (for $0 \leqslant m \leqslant k$). If $x^\mu \mapsto \psi(x)$ represents a local section s of $E \upharpoonright \mathscr{C}$, then the coordinates of $j^k s$ at "$b = (x^\mu)$" are given by $\psi_{\mu_1 \ldots \mu_m} = \partial_{\mu_1} \ldots \partial_{\mu_m} \psi(x)$, $0 \leqslant m \leqslant k$, where $\partial_\mu = \partial / \partial x^\mu$ are the partial derivatives. The projection $\pi_\ell^k : J_k(E) \to J_\ell(E)$

($\ell \leqslant k$) corresponds (on $\mathcal{O}$) to the canonical projections

$$(\psi_{\mu_1^k \dots \mu_k^k}, \dots, \psi_{\mu_1^\ell \dots \mu_\ell^\ell}, \dots, \psi, x^\mu) \mapsto (\psi_{\mu_1^\ell \dots \mu_\ell^\ell}, \dots, \psi, x^\mu)$$

of

$$\Big(\bigoplus_{m=0}^{m=k} S^m(\mathbb{R}^n)^* \otimes E_0\Big) \times \mathbb{R}^n \quad \text{onto} \quad \Big(\bigoplus_{m=0}^{m=\ell} S^m(\mathbb{R}^n)^* \otimes E_0\Big) \times \mathbb{R}^n .$$

If F is another vector bundle over B with trivialisation over $\mathcal{O}$, $F \restriction \mathcal{O} \simeq F_0 \times \mathcal{O} \simeq F_0 \times \mathbb{R}^n$, and if $L \in \mathcal{D}_k(E, F)$ is a partial differential operator, then L is represented over $\mathcal{O}$ by $L = \sum_{m=0}^{m=k} \sigma^{\mu_1 \dots \mu_m}(x)\, \partial_{\mu_1} \dots \partial_{\mu_m}$, where the $\sigma^{\mu_1 \dots \mu_m}(x)$ are linear mappings of E_0 in F_0 (i.e. elements of $F_0 \otimes E_0^*$); $p_k(L) : J_k(E) \to F$ corresponds to the mapping $(\psi_{\mu_1^k \dots \mu_k^k}, \dots, \psi, x^\mu) \mapsto \Big(\sum_{m=0}^{m=k} \sigma^{\mu_1 \dots \mu_m}(x)\, \psi_{\mu_1 \dots \mu_m}, x^\mu\Big)$, and the symbol $\sigma_k(L)$ is represented by the map

$$(\psi_{\mu_1 \dots \mu_k}, x^\mu) \mapsto (\sigma^{\mu_1 \dots \mu_k}(x)\, \psi_{\mu_1 \dots \mu_k}, x^\mu)$$

from $(S^k(\mathbb{R}^n)^* \otimes E_0) \times \mathbb{R}^n$ in $F_0 \times \mathbb{R}^n$.

In the previous example of a connection ∇ on E, F is $T^* \otimes E$ so, in the above coordinates, $T^* \otimes E \restriction \mathcal{O} \simeq (\mathbb{R}^n)^* \otimes E_0 \times \mathbb{R}^n$ and elements of $T^* \otimes E \restriction \mathcal{O}$ are represented by (ψ_ν, x^μ) where ψ_ν are n elements of E_0. Since $\sigma_1(\nabla)$ is the identity mapping of $T^* \otimes E$, it corresponds to $(\psi_\nu, x^\mu) \mapsto (\psi_\nu, x^\mu)$, so $p_1(\nabla) : J_1(E) \to T^* \otimes E$ is of the form $(\psi_\nu, \psi, x^\mu) \mapsto (\psi_\nu + A_\nu(x)\, \psi, x^\mu)$ where $A_\nu(x)$ are endomorphisms of E_0. Thus if $x \mapsto \psi(x)$ represents a local section s of E on $\mathcal{O}$, ∇s is represented by $x \mapsto \partial_\nu \psi(x) + A_\nu(x)\, \psi(x)$ in these coordinates. Furthermore the curvature Ω of ∇ corresponds to the 2-form $\frac{1}{2} F_{\mu\nu}(x)\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)].$$

3. LINEAR PARTIAL DIFFERENTIAL EQUATIONS

3.1. DEFINITIONS. Let E be a smooth vector bundle over B. A *regular k-th order linear partial differential equation on E* is a smooth sub-vector bundle R of $J_k(E)$; *a (local) solution of R* is a (local) section s of E such that $j^k s$ is a (local) section of R.

Let F be another vector bundle over B and let $L \in \mathcal{D}_k(E, F)$ be regular; then $\ker p_k(L)$ is a regular k-th order equation on E and $s \in \Gamma(E)$ is a solution of $\ker p_k(L)$ iff $\cdot\, Ls = 0$.

Now if R is a regular k-th order equation on E, then $J_k(E)/R$ is well defined as vector bundle over B and, if $r : J_k(E) \to J_k(E)/R$ is the canonical projection, then $r \circ j^k$ is in $\mathscr{D}_k(E, J_k(E)/R)$ with $\ker p_k(r \circ j^k) = R$ and if $L \in \mathscr{D}_k(E, F)$ is such that $\ker p_k(L) = R$, L factorized through $r \circ j^k$ and an injective vector bundle homomorphism $J_k(E)/R \to F$. Thus it is clear that if $R = \ker p_k(L)$, then $R^{(\ell)} = \ker p_{k+\ell}(j^\ell \circ L)\ (\subset J_{k+\ell}(E))$ does only depend on R; in fact we have:

$$R^{(\ell)} = J_\ell(R) \cap J_{k+\ell}(E) \quad (\text{in } J_\ell(J_k(E)))$$

$R^{(\ell)}$ is called *the ℓ-th prolongation of R* it is generally not a sub-bundle of $J_{k+\ell}(E)$ but only a family of subspaces of $J_{k+\ell}(E)$ over B. When, for all integers $\ell \geqslant 0$, $R^{(\ell)}$ is a vector bundle over B, the equation R is said to be *completely regular*.

3.2. Notations and remarks

1. One may define a (non-regular) k-th order linear partial differential equation on E to be a *family of subspaces of $J_k(E)$* over B; by this we mean a subset R of $J_k(E)$ which contains the zero section of $J_k(E)$ and which is such that, for any $b \in B$, $R_b = R \cap J_{b,k}(E)$ is a linear subspace of $J_{b,k}(E)$. A (local) solution of R is again in (local) section s of E satisfying $j_b^k s \in R_b$, for $b \in B$ (in its domain). Thus if R is a k-th order regular equation, then $R^{(\ell)}$ is a $(k + \ell)$-th order equation which is generally non-regular but which has the same (local) solutions as R.

2. If $L \in \mathscr{D}_k(E, F)$ is a differential operator, we shall speak of «the equation $Ls = 0$» to denote the equation $\ker p_k(L)$. One must be aware that different differential operators eventually correspond to the same equation.

3. It follows from the definition that, if R is a k-th order linear equation on E, we have $\pi_{k+\ell}^{k+\ell+m}(R^{(\ell+m)}) \subset R^{(\ell)}$, $\forall \ell, m \geqslant 0$, but, in general, $\pi_{k+\ell}^{k+\ell+m}$ is not surjective from $R^{(\ell+m)}$ to $R^{(\ell)}$ (i.e. the inclusion may be strict). (The next definition will avoid this non-surjectivity).

3.3. DEFINITION. A regular k-th order linear partial differential equation R on E is said to be *formally integrable* if it is completely regular and if for any $\ell \geqslant 0$ $\pi_{k+\ell}^{k+\ell+1}$ induces a surjective map of $R^{(\ell+1)}$ on $R^{(\ell)}$, (i.e. $\pi_{k+\ell}^{k+\ell+1}(R^{(\ell+1)}) = R^{(\ell)}$).

The origin of this terminology is the following. Let b be a point of $B\ (b \in B)$ and let us try to solve R by Taylor expansion at b. We may identify $R_b^{(\ell)}$ with the set of coefficients of Taylor expansions to order $k + \ell$ satsifying R to the corresponding order at b; indeed if s is a local section of E around b with $j_b^{k+\ell} s \in R_b^{(\ell)}$, then s satisfies R modulo sections vanishing to order $k + \ell + 1$ at b. But in order that each element u of $R_b^{(\ell)}$ could be interpreted as the Taylor expansion to order $k + \ell$ of some local solution s one has to assume that there is a $\tilde{u} \in R_b^{(\ell+1)}$ corresponding to Taylor expansion to order $k + \ell + 1$ of s, so since $\pi_{k+\ell}^{k+\ell+1} \circ j^{k+\ell+1} = j^{k+\ell}$, such that $\pi_{k+\ell}^{k+\ell+1} \tilde{u} = u$. Thus, apart from complete regularity which is natural in the framework of Taylor expansions, formal integrability is just the condition needed to formally solve at any order the equation at b by starting from any element of $R_b^{(\ell)}$ and this for any $\ell \geqslant 0$ and any $b \in B$.

Let R be a k-th order linear equation on E and suppose that R is formally integrable; this means that, given $\ell \geqslant 0$, $b \in B$ and $u \in R_b^{(\ell)}$, there are no algebraic obstructions to the existence of a solution s of R in a neighbourhood of b satisfying $j_b^{k+\ell} s = u$. Unfortunately, it is not always the case (although this is often true) at the «smooth level». This is, however, true at the «analytic level» as one may guess, (it is a consequence of Cartan - Kähler theorem). Let us make this precise. If B is an analytic manifold and E is an analytic vector bundle over B, a k-th order linear equation R on E is said *to be analytic* if R is an analytic subvector bundle of $J_k(E)$; *an analytic (local) solution* of R is an analytic (local) section of E which is solution of R. With this terminology the result is the following [11].

3.4. THEOREM. *Let R be a formally integrable analytic k-th order linear equation. Then, for any integer $\ell \geqslant 0$, for any $b \in B$ and for any $u \in R_b^{(\ell)}$, there is a local analytic solution s of R in a neighbourhood of b for which we have: $j_b^{k+\ell} s = u$.*

3.5. DEFINITION. Let R be a k-th order linear partial differential equation. Then $N = R \cap S^k T^* \otimes E$ is called the *symbol of* R; it is a family of subspaces of $S^k T^* \otimes E (\subset J_k(E))$ over B. Notice that if $R = \ker p_k(L)$, for $L \in \mathscr{D}_k(E, F)$, then $N = \ker \sigma_k(L)$. The symbol $N^{(\ell)}$ of $R^{(\ell)}$ does only depend on N since we have $N^{(\ell)} = S^\ell T^* \otimes N \cap S^{k+\ell} T^* \otimes E$, as easily seen; $N^{(\ell)}$ is called the *ℓ-th prolongation* of N.

We shall see that the obstructions to the formal integrability of R take their values in a family of spaces over B which does only depend on N.

3.6. PROPOSITION. [10] *Let R be a regular k-th order linear equation on E. Then, by using the canonical inclusion $J_k(E) \subset J_m(J_{k-m}(E))$, $1 \leqslant m \leqslant k$, R may be considered as a regular m-th order linear equation on $J_{k-m}(E)$; its ℓ--th prolongation is again $R^{(\ell)}$ considered as a $(\ell + m)$-order equation on $J_{k-m}(E)$,* (by using $J_{k+\ell}(E) \subset J_{m+\ell}(J_{k-m}(E))$). *Furthermore, for any solution s of R as k-th order equation on E, $j^{k-m} s$ is solution of R as m-th order equation on $J_{k-m}(E)$ and conversely, for any solution $\tilde{s}$ of R as m-th order equation* *on* $J_{k-m}(E)$, *there is a solution* s *of* R *as k-th order equation on* E *for which* $j^{k-m}s = s$. This proposition allows to replace any equation by a first order one: it is, in fact, a very old trick and it is easy to prove.

Another easy useful result is the following [11].

3.7. PROPOSITION. *Let* R *be a regular k-th order linear equation and suppose that its* ℓ*-th prolongation,* $R^{(\ell)}$, *is also regular. Then we have:* $(R^{(\ell)})^{(m)} = R^{(\ell+m)}$.

3.8. Example

Let ∇ be a connection on the vector bundle E over B, (see in 2.6), and let us consider the regular 1-th order linear equation $R = \ker p_1(\nabla)$ on E. The (local) solutions of R are the (local) sections s of E satisfying $\nabla s = 0$, they are called *horizontal (local) sections of* E *for* ∇. Since $\sigma_1(\nabla)$ is the identity mapping of $T^* \otimes E$ on itself, (see in 2.6), it follows that the symbol $\mathcal{N}$ of R is the zero section of $T^* \otimes E$: $\mathcal{N} = 0$. This implies that the symbol $\mathcal{N}^{(\ell)}$ of the ℓ-th prolongation $R^{(\ell)}$ of R also vanishes: $\mathcal{N}^{(\ell)} = 0$. It follows that $\pi_\ell^{\ell+1}$ induces an injective mapping of $R^{(\ell)}$ in $R^{(\ell-1)}$ for any $\ell \geqslant 1$, $(\mathcal{N}^{(\ell)} = 0 \to R^{(\ell)} \to R^{(\ell-1)})$; for $\ell = 0$, π_0^1 induces a bijection of R on E (since it is surjective as it follows from 2.6). In order that $\pi_{\ell+1}^{\ell+2}$ induces, for any $\ell \geqslant 0$, a surjective (and therefore bijective) mapping of $R^{(\ell+1)}$ on $R^{(\ell)}$ it is therefore necessary and sufficient that $\pi_0^{\ell+1}$ induces, for any $\ell \geqslant 0$, a surjection of $R^{(\ell)}$ on E; $\pi_0^{\ell+1}$ is then an isomomorphism of $R^{(\ell)}$ on E and $R^{(\ell)}$ is regular. Thus R is formally integrable if and only if $\pi_0^{\ell+1}(R^{(\ell)}) = E$, $\forall \ell \geqslant 0$.
Since $s \mapsto \Omega s$ factorizes through $s \to j^1 \circ \nabla s$ and a bundle homomorphism, it follows that $\pi_0^2(R^{(1)})$ is contained in (it is in fact equal to) the subset $\{\mathcal{G} \in E \mid \Omega \mathcal{G} = 0\}$ of E; so $\pi_0^2(R^{(1)}) = E$ implies $\Omega = 0$. Thus in order that R be formally integrable it is necessary that Ω vanishes. This is also sufficient because, as it is well known, if $\Omega = 0$ then E admits around any point of B local trivialisations $E \restriction \mathcal{C} \simeq E_0 \times \mathcal{C}$ in which ∇ corresponds to the usual differential of vector valued functions so, in these trivialisations, the horizontal sections are the constant E_0-valued functions on $\mathcal{C}$. This also shows that, in this case, formal integrability implies local resolubility in the sense of theorem 3.4 but without any analyticity assumption.

In some sense, formal integrability of linear equations are generalisations of zero curvature conditions and it is useful to known when a non-linear partial differential equation is exactly the formal integrability condition of a linear one because then its properties rely to properties of the corresponding linear system. We shall see that Einstein equations and Yang-Mills equations are non--linear equations of this type. Furthermore we shall show that pure Einstein equations and pure Yang-Mills equations may be written as zero-curvature conditions for connections on appropriate vector bundles.

3.9. DEFINITION. Let $N \subset S^k T^* \otimes E$ be a family of subspaces of $S^k T^* \otimes E$ over B, (see 3.2); N will be called *homogeneous* if, for any pair (b, b') of elements of B, there is (at last one) an isomorphism of vector spaces of T_b^* on $T_{b'}^*$ and (at last one) an isomorphism of vector spaces of E_b on $E_{b'}$ such that the correspondig isomorphism of $S^k T_b^* \otimes E_b$ on $S^k T_{b'}^* \otimes E_{b'}$ induces an isomorphism of N_b on $N_{b'}$; it then follows that the corresponding isomorphism of $S^{k+\ell} T_b^* \otimes E_b$ on $S^{k+\ell} T_{b'}^* \otimes E_{b'}$ induces an isomorphism of $N_b^{(\ell)}$ on $N_{b'}^{(\ell)}$, for any $\ell \geqslant 0$, where $N^{(\ell)} = S^\ell T^* \otimes N \cap S^{k+\ell} T^* \otimes E$. All the equations that will be considered in this paper have homogeneous symbol which implies, in particular, that the $N^{(\ell)}$ are vector sub-bundles of the $S^{k+\ell} T^* \otimes E \, (\ell \geqslant 0)$.

4. COMOLOGY OF SYMBOLS

4.1 In the section we describe spaces associated with symbols which will be useful for the analysis of formal integrability of linear parital differential equations on E. Since the constructions are pointwise on B, i.e. are carried over each point $b \in B$, it will be convenient to drop the label b and to consider that T, T^*, E etc. . . . are fixed finite dimensional vector spaces which will be later the fibres of (the corresponding) vector bundles over a point $b \in B$. Thus now T and E are finite dimensional vector spaces and we call *E-valued k-symbol on T* or simply *k-symbol* when no confusion arises a linear subspace N of $S^k T^* \otimes E$ where $S^k T^* \otimes E$ is the space of symmetric k-linear maps of T^k in E. The *ℓ-th prolongation $N^{(\ell)}$ of N* is the space of symmetric $(k+\ell)$-linear maps $\psi : T^{k+\ell} \to E$, (i.e. $\psi \in S^{k+\ell} T^* \otimes E$), such that

$$(v_1, \ldots, v_k) \mapsto \psi(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+\ell})$$

is an element of N for any $v_{k+1}, \ldots, v_{k+\ell} \in T$. In other words we have, (compare with 3.5):

$$N^{(\ell)} = S^\ell T^* \otimes N \cap S^{k+\ell} T^* \otimes E.$$

$N^{(\ell)}$ is a $(k+\ell)$-symbol and we have $(N^{(\ell)})^{(m)} = N^{(\ell+m)}$.

4.2 Let us define linear maps [9]

$$\delta : \Lambda^s T^* \otimes S^r T^* \to \Lambda^{s+1} T^* \otimes S^{r-1} T^*$$

by $\delta(\omega_1 \wedge \ldots \wedge \omega_s \otimes (\omega)^r) = \omega \wedge \omega_1 \wedge \ldots \wedge \omega_s \otimes (\omega)^{r-1}$ for $r \geqslant 1$. We write $\delta(\Lambda^s T^*) = 0$ for $r = 0$ and, more generally, we make the convention $S^{-r} T^* = 0$

for $r \geqslant 1$, $(s \geqslant 0)$. We clearly have $\delta^2 = 0$ so the sequences

$$0 \to S^{r+s}T^* \xrightarrow{\delta} T^* \otimes S^{r+s-1}T^* \to \ldots \xrightarrow{\delta} \Lambda^s T^* \otimes S^r T^* \to \ldots$$

are complexes of vector spaces. By tensor product with E, we obtain the complexes

$$0 \to S^{r+s}T^* \otimes E \xrightarrow{\delta} \ldots \xrightarrow{\delta} \Lambda^s T^* \otimes S^r T^* \otimes E \xrightarrow{\delta} \ldots$$

4.3. LEMMA. *The above sequences are exact for $r + s \geqslant 1$, i.e.* $\operatorname{Im}(\delta) = \operatorname{Ker}(\delta)$.
This is the formal Poincaré lemma for Taylor expansions at the origin of T of E-valued differential forms on T. Indeed an element of $\Lambda^s T^* \otimes S^r T^* \otimes E$ is canonically a E-valued s-form on T which is homogeneous of degree r as function on T, the operator δ corresponds to the exterior differential. Since the usual homotopy of forms corresponding to dilatation of T preserves the homogeneity, it provides a homotopy for the above sequences.
The s-form on T corresponding to $\omega_1 \wedge \ldots \wedge \omega_s \otimes (\omega)^r$ is

$$x \mapsto \frac{[\omega(x)]^r}{r!} \; \omega \wedge \ldots \wedge \omega_s.$$

4.4. DEFINITIONS. Let N be a E-valued k-symbol on T. We have by definition $\delta^{-1}(T^* \otimes N) = N^{(1)}$ and therefore $\delta^{-1}(T^* \otimes N^{(\ell)}) = N^{(\ell+1)} \subset S^{k+\ell+1}T^* \otimes E$. This implies in particular that we have $\delta(\Lambda^s T^* \otimes N^{(\ell)}) \subset \Lambda^{s+1}T^* \otimes N^{(\ell-1)}$, (by convention $N^{(0)} = N$), for $\ell \geqslant 1$. *The Spencer* [9] *cohomology of* N is the cohomology of the sequences, (called δ-*sequences of* N).

$$0 \to N^{(\ell)} \xrightarrow{\delta} T^* \otimes N^{(\ell-1)} \xrightarrow{\delta} \ldots \xrightarrow{\delta} \Lambda^s T^* \otimes N^{(\ell-s)} \xrightarrow{\delta} \ldots \xrightarrow{\delta} \Lambda^\ell T^* \otimes N$$
$$\xrightarrow{\delta} \Lambda^{\ell+1} T^* \otimes S^{k-1}T^* \otimes E.$$

The cohomology at $\Lambda^s T^* \otimes N^{(\ell-s)}$ of above sequence will be denoted by $H^{\ell-s+1,s}(N)$; thus

$$H^{r,s}(N) = \{\alpha \in \Lambda^s T^* \quad N^{(r-1)} \mid \delta\alpha = 0\} / \delta(\Lambda^{s-1}T^* \quad N^{(r)})$$

are defined for $r \geqslant 1$ and $s \geqslant 0$. N is said to be *p-acyclic* if $H^{r,s}(N) = 0$ for $0 \leqslant s \leqslant p$, $r \geqslant 1$; it is said to be *involutive* if $H^{r,s}(N) = 0$ for any $s \geqslant 0$ and $r \geqslant 1$, (i.e. if it is dim (T)-acyslic).

4.5. PROPOSITION. *Let N be a E-valued k-symbol on T and m be an integer with $1 \leqslant m \leqslant k$. Then, by using the canonical inclusion*
$S^k T^* \subset S^m T^* \otimes S^{k-m} T^* \otimes E$, *$N$ may be considered as a $S^{k-m}T^* \otimes E$-valued m-symbol on T; its ℓ-th prolongation is again $N^{(\ell)}$ considered as a $S^{k-m}T^* \otimes E$-*

-valued $(m+\ell)$-symbol on T, (by using $S^{k+\ell}T^* \subset S^{m+\ell}T^* \otimes S^{k-m}T^*$). Furthermore the Spencer cohomology of N is the same as the Spencer cohomology of N considered as a $S^{k-m}T^* \otimes E$-valued m-symbol on T.

This proposition follows easily from the definititions. The first part is the «symbolic» counterpart of proposition 3.6.

4.6. Remarks

1. One must be aware of the fact that, if N is a k-symbol, there is a shift of $k-1$ in r in our definition of $H^{r,s}(N)$ with respect to the one of other authors, the last proposition is the very reason for our convention.

2. It immediately follows from the definitions that we have $H^{r,0}(N)=0$ and $H^{r,1}(N)=0$, i.e. any symbol is 1-acyclic.

3. We have $H^{r,s}(N^{(\ell)}) = H^{r+\ell,s}(N)$ for any $r \geqslant 1$ and $s \geqslant 0$.

4. Up to now, if N is a k-symbol, $H^{r,p}(N)$ are defined only for $r \geqslant 1$, it will be convenient in the following to define $H^{0,p}(N)$ as the cohomology of

$$\Lambda^{p-1}T^* \otimes N \xrightarrow{\delta} \Lambda^p T^* \otimes S^{k-1}T^* \otimes E \xrightarrow{\delta} \Lambda^{p+1}T^* \otimes S^{k-2}T^* \otimes E$$

for $p \geqslant 1$ and $H^{0,0}(N) = \begin{cases} E & \text{for } k=1 \\ 0 & \text{for } k \geqslant 2 \end{cases}$.

Acyclicity and involutivity still refer to the $H^{r,s}(N)$ for $r \geqslant 1$.

4.7. DEFINITION. [17] Let $(e_1, \ldots, e_n)$ be an ordered basis of T. For any $F \subset S^rT^* \otimes E$, we denote by $F_m = F_{(e_1,\ldots,e_m)}$ the set of $\psi \in F$ such that $\psi_{(e_s, v_1,\ldots,v_{r-1})} = 0$, $\forall s \leqslant m$. $(e_1, \ldots, e_n)$ is said to be *quasi-tegular for* $N\ (\subset S^kT^* \otimes E)$ if $\dim N^{(1)} = \sum_{m=1}^{n-1} \dim N_m + \dim N$; this is equivalent to say that the maps $N_m^{(1)} \xrightarrow{\delta_{m+1}} N_m$ are surjective for $0 \leqslant m \leqslant n-1$, (where $(\delta_m \psi)(v_1 \ldots v_k) = \psi(e_m, v_1, \ldots, v_k)$), i.e. that we have exact sequences

$$0 \longrightarrow N_{m+1}^{(1)} \xrightarrow{\subset} N_m^{(1)} \xrightarrow{\delta_{m+1}} N_m \longrightarrow 0.$$

4.8. LEMMA. *If $(e_1, \ldots, e_n)$ is quasi-regular for N, then it is quasi-regular for $N^{(\ell)}$*, $(\forall \ell \geqslant 0)$. Proof by induction.

4.9. THEOREM. [17] *N is involutive if and only if it has a quasi-regular basis.*

The proof that existence of a quasi-regular basis implies involutivity is easy, by using lemma 4.8. The proof of the converse in more difficult. By using this theorem, one proves.

4.10. THEOREM. [10] *Let N be as above; there is an integer μ ($\geqslant 0$) depending only on k ($=$ order of N), $n = \dim T$ and $\dim E$ such that $N^{(\mu)}$ is involutive, i.e. such that we have*

$$H^{r,s}(N) = 0, \quad \forall r \geqslant \mu + 1, \quad (\forall s \geqslant 0).$$

4.11. Remark

Let ∂^k denote the partial derivatives of order k on T with respect to vector basis; if f is a smooth E-valued function on T, then $\partial^k f(x)$ is for any $x \in T$ an element of $S^k T^* \otimes E$. Consider, for a E-valued k-symbol N on T, the equation

$$\partial^k f(x) \in N, \quad \forall x \in T \qquad (\mathscr{E}_N).$$

Then $N^{(\ell)}$ may be identified to the set of solutions of ($\mathscr{E}_N$) which are homogeneous polynomials of degree $k + \ell$; therefore $\Lambda^p T^* \otimes N^{(\ell)}$ corresponds to E-valued p-forms on T with homogeneous coefficients of degree $k + \ell$ satisfying ($\mathscr{E}_N$) and δ corresponds to the usual exterior differential. Thus $\bigoplus_{\substack{r \geqslant 1 \\ s \geqslant 0}} H^{r,s}(N)$ is the cohomology of E-valued differential forms on T the coefficients of which are polynomials of order k (and of arbitrary degrees) satisfying ($\mathscr{E}_N$).

4.12. Examples

In all these examples (e_μ) $\mu = (1, \ldots, n)$ is a basis of T, the ∂_μ denote the partial derivatives $\partial/\partial x^\mu$ at the point $x = x^\mu e_\mu \in T$ and we identify tensorial objects on T, T^* with their components with respect to (e_μ) and the dual basis (writing for instance $x = (x^\mu)$).

1. Let E be $S^m T^* \otimes F$ where F is some finite dimensional vector space and let N be the E-valued k-symbol

$$N = S^{k+m} T^* \otimes F \subset S^k T^* \otimes (S^m T^* \otimes F);$$

then N is involutive by Lemma 4.3 and proposition 4.5. The corresponding equation ($\mathscr{E}_N$), (as above in 4.11), reads

$$\partial_{\mu_1} \ldots \partial_{\mu_{k-1}} (\partial_\alpha \psi_{\beta \nu_1 \ldots \nu_{m-1}}(x) - \partial_\beta \psi_{\alpha \nu_1 \ldots \nu_{m-1}}(x)) = 0.$$

2. *Exterior differential symbol.* Let us denote δ acting to the right by

$\delta' : S^r T^* \otimes \Lambda^s T^* \to S^{r-1} T^* \otimes \Lambda^{s+1} T^*$. Let $N_{[p]} \subset T^* \otimes \Lambda^p T^*$ denote the kernel of $\delta' : T^* \otimes \Lambda^p T^* \to \Lambda^{p+1} T^*$. $N_{[p]}$ is a $\Lambda^p T^*$-valued 1-symbol on T; its ℓ-th prolongation $N_{[p]}{}^{(\ell)}$ is the kernel of $\delta' : S^{\ell+1} T^* \otimes \Lambda^p T^* \to S^\ell T^* \otimes \Lambda^{p+1} T^*$. On the other hand $N_{[p+1]}^{(\ell)}$ is, by the formal Poincaré lemma 4.3, the image of $\delta' : S^{\ell+2} T^* \otimes \Lambda^p T^* \to S^{\ell+1} T^* \otimes \Lambda^{p+1} T^*$; thus we have short exact sequences

$$0 \to N_{[p]}^{(\ell+1)} \xrightarrow{\subset} S^{\ell+2} T^* \otimes \Lambda^p T^* \xrightarrow{\delta'} N_{[p+1]}^{(\ell)} \to 0$$

by tensorisation on the left by the $\Lambda^q T^*$ and using the fact that $\delta \circ \delta' = \delta' \circ \delta$ we obtain the short exact sequences of δ-complexes

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
0 & \longrightarrow & \Lambda^q T^* \otimes N_{[p]}^{(\ell+1)} & \longrightarrow & \Lambda^q T^* \otimes S^{\ell+2} T^* \otimes \Lambda^p T^* & \xrightarrow{\delta'} & \Lambda^q T^* \otimes N_{[p+1]}^{(\ell)} & \longrightarrow & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
0 & \longrightarrow & \Lambda^{q+1} T^* \otimes N_{[p]}^{(\ell)} & \longrightarrow & \Lambda^{q+1} T^* \otimes S^{\ell+1} T^* \otimes \Lambda^p T^* & \xrightarrow{\delta'} & \Lambda^{q+1} T^* \otimes N_{[p+1]}^{(\ell-1)} & \to & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

The complex in the middle has trivial cohomology (again by lemma 4.3) and therefore we have $H^{\ell+1,q}(N_{[p+1]}) \simeq H^{\ell+2,q+1}(N_{[p]})$.
Thus $H^{\ell+1,q}(N_{[p]}) \simeq H^{\ell+1+p,q+p}(N_{[0]})$, $\forall \ell, q, p \geqslant 0$. On the other hand, $N_{[0]} = \{0\} \subset T^* \otimes \mathbb{R} = T^*$, ($\delta' : T^* = T^* \otimes \mathbb{R} \to T^*$ is the identity of T^*), so $N_{[0]}^{(\ell)} = \{0\} \subset S^{\ell+1} T^* \otimes \mathbb{R} = S^{\ell+1} T^*$ and therefore
$H^{r,s}(N_{[p]}) \simeq H^{r+p,s+p}(N_{[0]}) = 0$, $\forall r \geqslant 1$, $\forall s, p \geqslant 0$; in other words $N_{[p]}$ is an involutive $\Lambda^p T^*$-valued 1-symbol, for any $p \geqslant 0$.
The corresponding equation $(\mathscr{E}_{N_{[p]}})$ reads: $\mathrm{d}\omega(x) = 0$ $x \mapsto \omega(x) \in \Lambda^p T^*$ is a p-form on T and d is the exterior differential.

3. «*Killing symbol*». Take $E = T^*$ and consider the 1-symbol N defined by $N = \Lambda^2 T^* \subset T^* \otimes T^*$.
One has $N^{(1)} = T^* \otimes \Lambda^2 T^* \cap S^2 T^* \otimes T^* = \{0\}$, so one has $N^{(\ell)} = \{0\}, \forall \ell \geqslant 1$. It follows that one has $H^{r,s}(N) = 0, \forall r \geqslant 2$ but $H^{1,2}(N)$ is non-trivial ($H^{1,2}(N) \neq 0$); thus this symbol is not involutive. The equation $(\mathscr{E}_N)$ reads: $(x \mapsto (\omega_\mu(x)) \in T^*)$

$$\partial_\mu \omega_\nu(x) + \partial_\nu \omega_\mu(x) = 0.$$

4. *Maxwell (spin 1) symbol.* Let g be a non degenerated symmetric bilinear form on T with matrice $(g_{\mu\nu}) = (g(e_\mu, e_\nu))$ and let $(g^{\mu\nu})$ be the inverse matrice. Let E be the space T^* and N be the 2-symbol

$$N = \{(A_{\lambda\mu,\nu}) \in S^2 T^* \otimes T^* \mid g^{\lambda\mu}(A_{\lambda\mu,\nu} - A_{\lambda\nu,\mu}) = 0\}.$$

One checks that a basis (e_μ) in which, for instance $(g_{\mu\nu})$ is diagonal, is quasi-regular for N. Thus N is involutive. The corresponding equation $(\mathscr{E}_N)$ reads,

$$g^{\lambda\mu}\partial_\lambda(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)) = 0.$$

5. *Linearized Einstein (spin 2) symbol.* [3, 5] g being as above, let E be the space $S^2 T^*$ and N be the 2-symbol

$$N = \{(h_{\lambda\rho,\mu\nu}) \in S^2 T^* \otimes S^2 T^* \mid g^{\lambda\mu}(h_{\lambda\rho,\mu\nu} + h_{\lambda\nu,\mu\rho} - h_{\lambda\mu,\rho\nu} - h_{\rho\nu,\lambda\mu}) = 0\}.$$

One checks again that a basis (e_μ) in which $(g_{\mu\nu})$ is diagonal is quasi-regular for N, so N is involutive and the equation $(\mathscr{E}_N)$ reads:

$$g^{\lambda\mu}\{\partial_\lambda(\partial_\rho h_{\mu\nu}(x) + \partial_\nu h_{\mu\rho}(x) - \partial_\mu h_{\rho\nu}(x)) - \partial_\rho\partial_\mu h_{\lambda\mu}(x)\} = 0.$$

6. *Dirac (spin 1/2) symbol.* g is as above and E_g denote a space of (irreducti-ble) representation of the Clifford algebra of g, i.e. a vector space such that we have a linear mapping $\gamma: T \to \mathrm{End}(E_g)$ satisfying $\gamma(x)\gamma(y) + \gamma(y)\gamma(x) = 2g(x,y)\mathbb{1}$ for any $x, y \in T$, γ_μ denotes $\gamma(e_\mu)$, so have $\gamma(x) = \gamma_\mu x^\mu$. Let $E = E_g$ and consider the 1-symbol N defined by

$$N = \{(\psi_\mu) \in T^* \quad E_g \mid g^{\lambda\mu}\gamma_\lambda \psi_\mu = 0\}.$$

One verifies again that any basis (e_μ) in which $(g_{\mu\nu})$ is diagonal is quasi-regular for N so N is involutive and the equation $(\mathscr{E}_N)$ reads:

$$g^{\lambda\mu}\gamma_\lambda \partial_\mu \psi(x) = 0.$$

7. *Rarita-Schwinger (spin 3/2) symbol.* [6] g, E_g, γ being as above, let E be the space $T^* \otimes E_g$ and N be the 1-symbol

$$N = \{(\psi_{\mu,\nu}) \in T^* \otimes (T^* \otimes E_g) \mid \epsilon^{\alpha_1 \ldots \alpha_{n-3}\mu\nu\lambda}\gamma_{\alpha_1} \cdots \gamma_{\alpha_{n-3}}\psi_{\mu,\nu} = 0\}$$

where $\epsilon^{\mu_1 \ldots \mu_n}$ is completely antisymmetric $((\epsilon^{\mu_1 \ldots \mu_n}) \in \Lambda^n T)$ with $\epsilon^{1,2,\ldots,n} = 1$. Again, this is an involutive symbol because any basis in which $(g_{\mu\nu})$ is diagonal is quasi-regular for N. The equation $(\mathscr{E}_N)$ reads: $\epsilon^{\alpha_1 \ldots \alpha_{n-3}\mu\nu\lambda}\gamma_{\alpha_1} \ldots \gamma_{\alpha_{n-3}}\partial_\mu \psi_\nu(x) = 0$, or, by, introducing $\tilde{\psi}(x) = \psi_\mu(x)\mathrm{d}x^\mu$, $\tilde{\gamma} = \gamma_\mu \mathrm{d}x^\mu$, and the exterior differential d,

$$\underbrace{\tilde{\gamma} \wedge \ldots \wedge \tilde{\gamma}}_{n-3} \wedge \mathrm{d}\,\psi(x) = 0.$$

4.13. Remarks

1. The dimensional computations in the examples 4.12 - 4, 5, 6, 7 to verify the quasi-regularity of (e_μ) when $(g_{\mu\nu})$ is diagonal are straightforward although tiedous. They are simplified in such basis (where $g_{\mu\nu}$ is diagonal); this does not mean that other basis are not quasi-regular, (I simply verified that it was O.K. for such basis).

2. Let E' and E'' be two vector spaces and let N', (resp. N''), be a E'-valued, (resp. E''-valued), k-symbol on T, then we have:

a) $N' \oplus N''$ is a $E' \oplus E''$-valued k-symbol on T and
$H^{r,s}(N' \oplus N'') = H^{r,s}(N') \oplus H^{r,s}(N'')$;

b) $N' \otimes E''$ is a $E' \otimes E''$-valued k-symbol on T and
$H^{r,s}(N' \otimes E'') = H^{r,s}(N') \otimes E''$.

3. Notice the following:

a) The T^*-valued 2-symbol N on T of example 4.12 - 4 contains the 2-symbol S^3T^*, i.e. $S^3T^* \subset N \subset S^2T^* \otimes T^*$,

b) the S^2T^*-valued 2-symbol N on T of example 4.12 - 5 contains the image of $S^3T^* \otimes T^*$ in $S^2T^* \otimes S^2T^*$ under $\delta : S^3T^* \otimes T^* \to S^2T^* \otimes S^2T^*$
$\delta((\omega_{\alpha\beta\delta,\gamma}))_{\mu\nu,\lambda\rho} = \omega_{\mu\nu\lambda,\rho} + \omega_{\mu\nu\rho,\lambda}$, i.e. $\delta(S^3T^* \otimes T^*) \subset N \subset S^2T^* \otimes S^2T^*$.

c) the $T^* \otimes E_g$-valued 1-symbol N of example 4.12 - 7 contains the 1-symbol $S^2T^* \otimes E_g \subset T^* \otimes T^* \otimes E_g$, i.e. $S^2T^* \otimes E_g \subset N \subset T^* \otimes T^* \otimes E_g$.

These properties are connected with gauge invariance [3] of corresponding equations and may be used to give another proof of the involutivity in these cases.

4. By their very definition, (see in 4.11) equations of the form $(\mathcal{E}_N)$ are formally integrable equations.

5. *Scalar symbols.* Let $N \subset S^kT^*$ be of codimension 1, i.e. $N = \ker \sigma$ for a linear form $\sigma \in S^kT$ on S^kT^* with $\sigma \neq 0$. Then there is a $\omega \in T^*$ such that $\sigma(\omega^k) \neq 0$ so $S^kT^* = N \oplus \mathbb{C} \cdot \omega^k$ and, if (e_μ) is a basis of T satisfying $\langle e_\mu, \omega\rangle = 0$ for $\mu = 1, \ldots, n-1$ ($\Rightarrow \langle e_n, \omega\rangle \neq 0$), (e_μ) is quasi-regular for N; this follows from the fact that the composition $S^{k+1}T^* \xrightarrow{\delta} T^* \otimes S^kT^* \xrightarrow{\mathrm{Id} \otimes \sigma} T^*$ is surjective so $\dim(S^{k+1}T^*) = \dim(N^{(1)}) + n$. Furthermore, the same argument shows that we have $\delta(S^{k+1}T^*) + T^* \otimes N = T^* \otimes S^kT^*$ and therefore $H^{0,2}(N) = 0$, so $H^{r,2}(N) = 0, \forall r \geqslant 0$.

5. INTEGRABILITY CRITERIA

Let E be a (smooth) vector bundle over B. Then we have the following result.

5.1. PROPOSITION. [9] *There is a unique linear mapping* $D : \Gamma(J_{k+1}(E)) \to \Gamma(T^* \otimes J_k(E))$ *satisfying*

a) $D(f \cdot s) = \mathrm{d}f \otimes \pi_k^{k+1}(s) + f D s$, *for any* $f \in C^\infty(B)$ *and* $s \in \Gamma(J_{k+1}(E))$.

b) $D \circ j^{k+1} = 0$.

The uniqueness is clear since if D' is another mapping with these properties, $D - D'$ is by a) $C^\infty(B)$-linear (i.e. it is a vector bundle homomorphism) and by b) it vanishes on $\mathrm{Im}\, j^{k+1}$ which spans $J_{k+1}(E)$ so $D - D' = 0$. For the existence, we choose coordinates (see in 2.7) and write $Ds = \mathrm{d}x^\mu D_\mu s(x)$ with $(D_\mu s(x))\, \mu_1 \ldots \mu_m = \partial_\mu s_{\mu_1 \ldots \mu_m}(x) - s_{\mu\mu_1 \ldots \mu_m}(x)$, $\forall m \leqslant k$; i.e. Ds is the pull back by s of the canonical $J_k(E)$-valued one form θ on $J_{k+1}(E)$, $Ds = s^*\theta$.

By a), D is a 1-th order differential operator from $J_{k+1}(E)$ in $T^* \otimes J_k(E)$ with symbol $\sigma_1(D) = \mathrm{Id} \otimes \pi_k^{k+1} : T^* \otimes J_{k+1}(E) \to T^* \otimes J_k(E)$. $Ds = 0$ is equivalent to $s = j^{k+1} s_0$ for some local section s_0 of E.

One extends D to $\Gamma(\Lambda T^* \otimes J(E))$ by imposing

$$D\, \omega \otimes s = \mathrm{d}\, \omega \otimes \pi_k^{k+1}(s) + (-1)^p \omega \wedge Ds \quad \text{for} \quad \omega \in \Gamma(\Lambda^p T^*).$$

$s \in \Gamma(J_{k+1}(E))$, $\forall p, s \geqslant 0$ and $D\, \Gamma(E) = 0$. We then have:

$$D\, \Gamma(\Lambda^p T^* \otimes J_{k+1}(E)) \subset \Gamma(\Lambda^{p+1} T^* \otimes J_k(E)) \quad \text{and} \quad D^2 = 0.$$

So we have the complexes

$$0 \longrightarrow \Gamma(E) \xrightarrow{j^{r+s}} \Gamma(J_{r+s}(E)) \xrightarrow{D} \ldots \xrightarrow{D} \Gamma(\Lambda^s T^* \otimes J_r(E)) \xrightarrow{D} \ldots (\mathscr{C}_{r+s})$$

5.2. LEMMA. *The above sequences are exact.*

Notice that if $s \in \Gamma(S^{k+1} T^* \otimes E)$ then $Ds = -\delta s \in \Gamma(T^* \otimes S^k T^* \otimes E)$. Consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & 0 \to & \Gamma(S^{r+s+1}T^* \otimes E) & \xrightarrow{-\delta} & \cdots & \xrightarrow{-\delta} & \Gamma(\Lambda^s T^* \otimes S^{r+1}T^* \otimes E) & \xrightarrow{-\delta} & \cdots \\
 & \downarrow & \downarrow & & & & \downarrow & & \\
0 \to \Gamma(E) & \xrightarrow{j^{r+s+1}} & \Gamma(J_{r+s+1}(E)) & \xrightarrow{D} & \cdots & \xrightarrow{D} & \Gamma(\Lambda^s T^* \otimes J_{r+1}(E)) & \xrightarrow{D} & \cdots \\
\downarrow \text{id} & & \downarrow \pi^{r+s+1}_{r+s} & & & & \downarrow \text{id} \otimes \pi^{r+1}_{r} & & \\
0 \to \Gamma(E) & \xrightarrow{j^{r+s}} & \Gamma(J_{r+s}(E)) & \xrightarrow{D} & \cdots & \xrightarrow{D} & \Gamma(\Lambda^s T^* \otimes J_r(E)) & \xrightarrow{D} & \cdots \\
\downarrow & & \downarrow & & & & \downarrow & & \\
0 & & 0 & & & & 0 & &
\end{array}$$

This diagram is commutative and the columns are exact so we have a short exact sequence of complexes

$$0 \longrightarrow \mathscr{C}_{r+s+1}{}^{(0)} \longrightarrow \mathscr{C}_{r+s+1} \longrightarrow \mathscr{C}_{r+s} \longrightarrow 0.$$

By 4.3 we know that the cohomology of $\mathscr{C}_{r+s+1}{}^{(0)}$ vanishes which implies that the cohomology of $\mathscr{C}_{r+s+1}$ is the same as the one of $\mathscr{C}_{r+s}$ and therefore, it is the same as the cohomology of $(\mathscr{C}_0)$

$$0 \longrightarrow \Gamma(E) \xrightarrow{\text{id} = j^0} \Gamma(E) \longrightarrow 0 \quad \text{which obviously vanishes.}$$

5.3. LEMMA. [14] *Let R be a regular k-th order linear partial differential equation on E and suppose that its ℓ-th prolongation, $R^{(\ell)}$, is also regular. Then, a section s of $J_{k+\ell+1}(E)$ takes its values in $R^{(\ell+1)}$, (i.e. $s(b) \in R_b^{(\ell+1)}$), if and only if we have: $\pi^{k+\ell+1}_{k+\ell} \circ s(b) \in R_b^{(\ell)}$ and $Ds(b) \in T_b^* \otimes R_b^{(\ell)}$, for any $b \in B$.*

Consider the following diagram

$$\begin{array}{ccccc}
 & & J_1(J_{k+\ell}(E)) & & \\
 & \overset{\lambda}{\nearrow} & & \overset{\pi^1_0}{\searrow} & \\
J_1(J_{k+\ell+1}(E)) & & & & J_{k+\ell}(E) \\
 & \underset{\mu}{\searrow} & & \underset{\pi^{k+\ell+1}_{k+\ell}}{\nearrow} & \\
 & & J_{k+\ell+1}(E) & &
\end{array}$$

where $\lambda = J_1(\pi^{k+\ell+1}_{k+\ell})$, (see in 2.3 - 1), and $\mu = \pi^1_0 : J_1(J_{k+\ell+1}(E)) \to J_{k+\ell+1}(E)$. This diagram is commutative and $\pi^{k+\ell+1}_{k+\ell}$ is the restriction of $\pi^1_0 : J_1(J_{k+\ell}(E)) \to J_{k+\ell}(E)$ to $J_{k+\ell+1}(E) \subset J_1(J_{k+\ell}(E))$. It follows that we have

$\pi_0^1 \circ (\lambda - \mu) = 0$ and therefore that $\lambda - \mu$ maps $J_1(J_{k+\ell+1}(E))$ into $T^* \otimes J_{k+\ell}(E)$. By using, for instance, the coordinates expression of D given in 5.1, we see that we have $\lambda - \mu = p_1(D) : J_1(J_{k+\ell+1}(E)) \to T^* \otimes J_{k+\ell}(E)$. The lemma then follows from this and from the fact that we have, by proposition 3.7:
$R^{(\ell+1)} = J_1(R^{(\ell)}) \cap J_{k+\ell+1}(E) = (R^{(\ell)})^{(1)}$.

5.4. Let R be a regular k-th order linear partial differential equation on E (with $k \geqslant 1$) and assume that the symbol N of R is homogeneous (see in 3.9). Then the $N^{(\ell)}$ are sub-vector-bundles of the $S^{k+\ell}T^* \otimes E$ and the mappings $\delta_b^{\ell,p} = \delta : \Lambda^p T_b^* \otimes N_b^{(\ell)} \to \Lambda^{p+1} T_b^* \otimes N_b^{(\ell-1)}$ have constant rank in $b \in B$, for $\ell \geqslant 0$, where we make the convention $N^{(-1)} = S^{k-1}T^* \otimes E$. it follows that $Z^{\ell+1,p}(N_b) = \ker(\delta_b^{\ell,p})$, $B^{\ell,p+1}(N_b) = \operatorname{Im}(\delta_b^{\ell,p})$ are the fibers of vector bundles and that the same is true for $H^{\ell,p}(N_b) = Z^{\ell,p}(N_b)/B^{\ell,p}(N_b)$ for any $\ell, p \geqslant 0$. (the $N^{0,p}(N_b)$ being defined in remark 4.6 - 4). We denote by $Z^{r,s}(N)$, $B^{r,s}(N)$ and $H^{r,s}(N)$ the corresponding vector bundles over B. Although we are here interested in equations with homogeneous symbols we come back to the general case to formulate the next general results.

5.5. In the following, if R is a regular k-th order linear equation, we shall denote by $\Gamma(R^{(\ell)})$, (resp. $\Gamma(N^{(\ell)})$), the subspace of $\Gamma(J_{k+\ell}(E))$, (resp. of $\Gamma(S^{k+\ell}T^* \otimes E)$), which consists of $R^{(\ell)}$-valued sections, (resp. of $N^{(\ell)}$-valued sections), and we make the convention $R^{(-1)} = J_{k-1}(E)$ and $N^{(-1)} = S^{k-1}T^* \otimes E$; k is always assumed to be greater or equal to 1.

$\Gamma(R^{(\ell)})$ and $\Gamma(N^{(\ell)})$ are $C^\infty(B)$-modules (for $\ell \geqslant -1$) and we define similarity $\Gamma(\Lambda^p T^* \otimes R^{(\ell)})$ and $\Gamma(\Lambda^p T^* \otimes N^{(\ell)})$, i.e.

$$\Gamma(\Lambda^p T^* \otimes R^{(\ell)}) = \Gamma(\Lambda^p T^*) \underset{C^\infty(B)}{\otimes} \Gamma(R^{(\ell)}) \quad \text{and}$$

$$(\Lambda^p T^* \otimes N^{(\ell)}) = \Gamma(\Lambda^p T^*) \underset{C^\infty(B)}{\otimes} \Gamma(N^{(\ell)}), \quad (p \geqslant 0,\ \ell \geqslant -1).$$

We have $D\Gamma(\Lambda^p T^* \otimes R^{(\ell+1)}) \subset \Gamma(\Lambda^{p+1} T^* \otimes R^{(\ell)})$ and the
$\delta : \Lambda^p T_b^* \otimes N_b^{(\ell+1)} \to \Lambda^{p+1} T_b^* \otimes N_b^{(\ell)}$ $(b \in B)$, defined as in section 4, induce homomorphisms of $C^\infty(B)$-modules again denoted by
$\delta : \Gamma(\Lambda^p T^* \otimes N^{(\ell+1)}) \to \Gamma(\Lambda^{p+1} T^* \otimes N^{(\ell)})$ satisfying $\delta^2 = 0$. We denote by $\Gamma H^{r,s}(N)$ the $C^\infty(B)$-module

$$\Gamma H^{r,s}(N) = \{u \in \Gamma(\Lambda^s T^* \otimes N^{(r-1)}) \mid \delta u = 0\} / \delta\Gamma(\Lambda^{s-1} T^* \otimes N^{(r)})$$

for $r + s \geqslant 1$ (we put $\Lambda^{-1}T^* = 0$) and $\Gamma H^{0,0}(N) = \Gamma(E)$ if $k = 1$ and $\Gamma H^{0,0}(N) = 0$ if $k \geqslant 2$; (compare with 4.6 - 4). When $Z^{r,s}(N_b)$ and $B^{r,s}(N_b)$ are fibers of vector bundles, $b \in B$, (for instance when N is homogeneous) the $H^{r,s}(N_b)$ are also fibers of a vector bundle denoted by $H^{r,s}(N)$ and $\Gamma H^{r,s}(N)$

is the $C^\infty(B)$-module of its sections i.e. $\Gamma H^{r,s}(N) = \Gamma(H^{r,s}(N))$. We also denote by $\pi^{k+\ell+1}_{k+\ell}$, or simply by π when no confusion arises, the homomorphism of $C^\infty(B)$-modules of $\Gamma(\Lambda^p T^* \otimes R^{(\ell+1)})$ in $\Gamma(\Lambda^p T^* \otimes R^{(\ell)})$ induced by $\pi^{k+\ell+1}_{k+\ell} : R^{(\ell+1)} \to R^{(\ell)}$.

5.6. PROPOSITION. *Let R be a regular k-th order linear partial differential equation on E, ($k \geq 1$). Assume that $R^{(\ell-1)}$ and $N^{(\ell)}$ are vector bundles and that $\pi^{k+\ell}_{k+\ell-1} : R^{(\ell)} \to R^{(\ell-1)}$ is surjective for some $\ell \geq 0$. Then $R^{(\ell)}$ is a vector bundle and there is a homomorphism of $C^\infty(B)$-modules $\varkappa_\ell : \Gamma(R^{(\ell)}) \to \Gamma H^{\ell,2}(N)$ for which we have: $\pi^{k+\ell+1}_{k+\ell}(\Gamma(R^{(\ell+1)})) = \ker(\varkappa_\ell)$.*

The kernel of $\pi^{k+\ell}_{k+\ell-1} : R^{(\ell)}_b \to R^{(\ell-1)}_b$ is $N^{(\ell)}_b$, for any $b \in B$, and therefore $R^{(\ell)}_b$ is (by the assumption) isomorphic to $N^{(\ell)}_b \oplus R^{(\ell-1)}_b$ which is the fibre of a vector bundle; therefore $\dim R^{(\ell)}_b$ is independent of $b \in B$ which implies that $R^{(\ell)}$ is a vector bundle. We now proceed to the construction of $\varkappa_\ell$.

Let s be in $\Gamma(R^{(\ell)})$; then, from the surjectivity of $\pi^{k+\ell}_{k+\ell-1}$ we have $Ds = \pi^{k+\ell}_{k+\ell-1}(u)$ where $u \in \Gamma(T^* \otimes R^{(\ell)})$ is unique up to element of $\Gamma(T^* \otimes N^{(\ell)})$; $Du \in \Gamma(\Lambda^2 T^* \otimes R^{(\ell-1)})$ is unique up to $\delta\Gamma(T^* \otimes N^{(\ell-1)})$ $(= -D\Gamma(T^* \otimes N^{(\ell)}))$ and is in fact in $\Gamma(\Lambda^2 T^* \otimes N^{(\ell-1)})$ since $\pi Du = D^2 s = 0$. Furthermore $\delta Du = -D^2 u = 0$ and therefore $\varkappa_\ell(s) = Du + \delta\Gamma(T^* \otimes N^{(\ell)})$ is a well defined element of $\Gamma H^{\ell,2}(N)$. Let f be an arbitrary function $f \in C^\infty(B)$; then

$$D(f \cdot s) = fDs + \mathrm{d}f \otimes \pi(s) = f\pi(u) + \pi(\mathrm{d}f \otimes s) = \pi(fu + \mathrm{d}f \otimes s)$$

and $D(fu + \mathrm{d}f \otimes s) = fDu + \mathrm{d}f \otimes \pi(u) - \mathrm{d}f \otimes Ds = fDu$ where $Ds = \pi(u)$ as above . It follows that

$$\varkappa_\ell(fs) = fDu + \delta\Gamma(T^* \otimes N^{(\ell)}) = f(Du + \delta\Gamma(T^* \otimes N^{(\ell)})) = f\,\varkappa_\ell(s)$$

so $\varkappa_\ell$ is a homomorphism of $C^\infty(B)$-modules. If we have $s = \pi(\tilde{s})$, $\tilde{s} \in \Gamma(R^{(\ell+1)})$, then we can take $u = D\tilde{s}$ above, which implies $Du = 0$ and therefore $\varkappa_\ell(s) = 0$, so we have $\pi^{k+\ell+1}_{k+\ell}(\Gamma(R^{(\ell+1)})) \subset \ker(\varkappa_\ell)$. If $\varkappa_\ell(s) = 0$ then we can choose u such that $Du = 0$ above so $u = Dv$ for some $v \in \Gamma(J_{k+\ell+1}(E))$ (by Lemma 5.2) and therefore $Ds = \pi(u) = D\pi(v)$; so (again by 5.2) we have $\pi(v) - s = j^{k+\ell}s_0$ where $s_0 \in \Gamma(E)$. Define $\tilde{s} \in \Gamma(J_{k+\ell+1}(E))$ by $\tilde{s} = v - j^{k+\ell+1}s_0$; we have $D\tilde{s} = Dv = u \in \Gamma(T^* \otimes R^{(\ell)})$ and $\pi(s) = \pi(v) - j^{k+\ell}s_0 = s \in \Gamma(R^{(\ell)})$ which, by lemma 5.3, implies $\tilde{s} \in \Gamma(R^{(\ell+1)})$. So we have $\pi^{k+\ell+1}_{k+\ell}(\Gamma(R^{(\ell+1)})) \supset \ker(\varkappa_\ell)$ and therefore the equality $\pi^{k+\ell+1}_{k+\ell}(\Gamma(R^{(\ell+1)})) = \ker(\varkappa_\ell)$.

5.7. COROLLARY. [10, 11, 14] *Let R be a regular k-th order linear partial differential equation on E. Assume that N and $N^{(1)}$ are vector bundles and that $H^{r,2}(N_b) = 0$ for any $r \geq 1$ and for any $b \in B$. Then R is formally integrable if and only if we have: $\pi^{k+1}_k(R^{(1)}) = R$.*

The necessity of the condition follows from the definition. By the same argument as in the beginning of the proof of 5.6, $\pi_k^{k+1}(R^{(1)}) = R$ implies that $R^{(1)}$ is a vector bundle. Assume that $N^{(\ell)}$ are vector bundles for $\ell \leqslant m$ and consider for each $b \in B$ the sequence

$$0 \to N_b^{(m+1)} \xrightarrow{\delta} T_b^* \otimes N_b^{(m)} \xrightarrow{\delta} \Lambda^2 T_b^* \otimes N_b^{(m-1)} \xrightarrow{\delta} \Lambda^3 T_b^* \otimes N_b^{(m-2)} \to \cdots$$
$$\cdots \to \Lambda^3 T_b^* \otimes N_b^{(m-2)} / \delta(\Lambda^2 T_b^* \otimes N_b^{(m-1)}) \to 0;$$

This sequence is exact so we have

$$\dim N_b^{(m+1)} + \dim(\Lambda^3 T_b^* \otimes N_b^{(m-2)} / \delta(\Lambda^2 T_b^* \otimes N_b^{(m-1)})) = C^{te}$$

independent of b so $\dim N_b^{(m+1)} = C^{te}$ independent of b since both terms are upper semicontinuous and therefore $N^{(m+1)}$ is also a vector bundle. Thus all $N^{(\ell)}$ are vector bundles. It follows that

$$\Gamma(T^* \otimes N^{(m)}) \xrightarrow{\delta} \Gamma(\Lambda^2 T^* \otimes N^{(m-1)}) \xrightarrow{\delta} \Gamma(\Lambda^3 T^* \otimes N^{(m-2)})$$

are also exact for $m \geqslant 1$ and thus $\Gamma H^{r,2}(N) = 0$ for $r \geqslant 1$. Assume now that $R^{(\ell)}$ are vector bundles for $0 \leqslant \ell \leqslant m$ and that $\pi_{k+\ell}^{k+\ell+1}(R^{(\ell+1)}) = R^{(\ell)}$ for $0 \leqslant \ell \leqslant m-1$. Then by 5.6, $\pi_{k+m}^{k+m+1}(\Gamma(R^{(m+1)})) = \Gamma(R^{(m)})$ which implies that $\pi_{k+m}^{k+m+1}(R^{(m+1)}) = R^{(m)}$ and that $R^{(m+1)}$ is a vector bundle since $N^{(m+1)}$ is a vector bundle. So $\pi_k^{k+1}(R^{(1)}) = R$ implies formal integrability.

5.8. COROLLARY. *Let E be vector bundle equipped with a connexion ∇ and let $\sigma : T^* \otimes E \to F$ be a vector bundle homomorphism such that its kernel N is a vector bundle. Let $R = \ker p_1(\sigma \circ \nabla) \subset J_1(E)$. R is a regular 1-th order equation with N as symbol and $\pi_1^2(\Gamma(R^{(1)})) = \Gamma(R)$ if and only if $\nabla\Gamma(N)$ and $\Omega\Gamma(E)$ are contained in $\delta\Gamma(T^* \otimes N)$*, ($\Omega$ is the curvature of ∇).

We have $\pi_0^1(R) = E$ and $p_1(\nabla) \oplus \pi_0^1$ is an isomorphism of vector bundles of $J_1(E)$ on $(T^* \otimes E) \oplus E$ (see in 2.6): under this isomorphism R is mapped on $N \oplus E$ and the mapping $\varkappa_0$ of 5.6 (the conditions of 5.6 are satisfied for $\ell = 0$, $k = 1$) corresponds to mappings $\varkappa_0^1 : \Gamma(N) \to \Gamma H^{0,2}(N)$ and $\varkappa_0^0 : \Gamma(E) \to \Gamma H^{0,2}(N)$ which must vanish whenever $\pi_1^2(\Gamma(R^{(1)}) = \Gamma(R)$ by 5.6. Now $D : \Gamma(J_1(E)) \to \Gamma(T^* \otimes E)$ corresponds to

$$(-\mathrm{Id}) \oplus \nabla : \Gamma(T^* \otimes E) \oplus \Gamma(E) \to \Gamma(T^* \otimes E)$$

under the isomorphism and so starting with $s \in \Gamma(R)$ corresponding to $\omega_1 \oplus \omega_0 \in \Gamma(T^* \otimes E) \oplus \Gamma(E)$ $Ds = \nabla\omega_0 - \omega_1$ and we can choose, (see in the proof of 5.6), $u \in \Gamma(T^* \otimes R) \simeq \Gamma(T^* \otimes T^* \otimes E) \oplus \Gamma(T^* \otimes E)$ with $\pi(u) = Ds$ to be itself represented by $\nabla\omega_0 - \omega_1$ under this isomorphism and therefore

$Du = \nabla^2 \omega_0 - \nabla \omega_1 = \Omega \omega_0 - \nabla \omega_1 \in \Gamma(\Lambda^2 T^* \otimes E)$ $(= \Gamma(Z^{0,2}(N)))$. Thus we have $\varkappa_0^0(\omega_0) = \Omega \omega_0 + \delta \Gamma(T^* \otimes N)$ and $\varkappa_0^1(\omega_1) = -\nabla \omega_1 + \delta \Gamma(T^* \otimes N)$ from which 5.8 follows.

5.9. Remarks

1. All the mappings involved above are local, so one may replace B in the above statements by any open set $\mathcal{O} \subset B$ and finally write the corresponding results at the level of sheaves theory for germs of sections.

2. The $H^{r,2}(N_b)$ are generally not easy to compute and it is hard to know whether they vanish; it is why involutivity is important because it can be checked more easily by applying lemma 4.8. There exist however 2-acyclic symbols which are not involutive [18].

5.10. Let R be a regular k-th order linear equation on E; R will be said to be *transitive* whenever $\pi^k_{k-1} : R \longrightarrow S^{k-1} T^* \otimes E$ is a surjection, (i.e. $\pi^k_{k-1}(R) = S^{k-1} T^* \otimes E$). For instance, regular 1-th order transitive equations on E are typically of the form $R = \ker p_1(\sigma \circ \nabla)$, as in 5.8, for some connexion ∇ on E and some $\sigma : T^* \otimes E \to F$ such that $N = \ker \sigma$ is a vector bundle and the first obstruction to formal integrability is $\varkappa_0(R) \in H^{0,2}(N)$; the corollary 5.8 just gives a convenient way of representing $\varkappa_0(R) = 0$.

In the following we shall be interested in transitive regular k-th order linear equations with homogeneous symbols so let R be such an equation. As pointed out in 5.4, the $H^{r,s}(N_b)$, (defined as in section 4), are the fibres of vector bundles $H^{r,s}(N)$ and the $\Gamma H^{r,s}(N)$ are the $C^\infty(B)$-modules of sections of the $H^{r,s}(N)$; it follows that under the conditions of proposition 5.6, $\varkappa_\ell$ gives, (since it is $C^\infty(B)$-linear), a vector bundle homomorphism again denoted by $\varkappa_\ell : R^{(\ell)} \to H^{\ell,2}(N)$, which must vanish in order that $\pi : R^{(\ell+1)} \to R^{(\ell)}$ be surjective; then, $R^{(\ell+1)}$ is a vector bundle and we have $\varkappa_{\ell+1} : R^{(\ell+1)} \to H^{\ell+1,2}(N)$ and so on . . . Since we know, by theorem 4.10, that $H^{m,2}(N) = 0$ for $m \geqslant \mu$ where μ only depends on dim (B) and rank (E), the formal integrability conditions of R read:

$$
\begin{array}{ll}
I_0(R) = \varkappa_0(R) = 0 & \\
I_1(R) = \varkappa_1(R^{(1)}) = 0 & \\
\dots\dots\dots\dots & (\mathcal{T}(R)) \\
I_\ell(R) = \varkappa_\ell(R^{(\ell)}) = 0 & \\
\dots\dots\dots\dots & \\
I_{\mu-1}(R) = \varkappa_{\mu-1}(R^{(\mu-1)}) = 0 &
\end{array}
$$

which is a finite system of non-linear partial differential equations for the «coefficients» of R, (i.e. for map $J_k(E) \to J_k(E)/R$). Thus the linear system R is formally integrable iff. $\mathcal{T}(R)$ is a satisfied and this may help for studying $\mathcal{T}(R)$ itself. When N is 2-acyclic, (for instance when it is involutive), $\mathcal{T}(R)$ reduces to $\varkappa_0(R) = 0$.

5.11. Examples

1. Let $b \mapsto g(b) \in S^2 T_b^*$ be a pseudo-metric on B; a a vector field X which generates infinitesimal isometries of (B, g) is called a *Killing vector field*. X is a Killing vector iff. $L_X g = 0$, where L denotes the Lie derivative. Let ω_X be the 1-form associated to X by g, i.e. $\omega_X(Y) = g(X, Y)$. In a local chart $(\mathcal{O}, x^\lambda)$, $L_X g = 0$ reads $\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0$, where X_λ are the components of ω_X, («covariant components» of X) and ∇ is the Levi-Civita connection. By applying 5.8 one readily sees that if R «is the equation $\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0$», i.e. $R = \ker p_1(\sigma \circ \nabla) \subset J_1(T^*)$ with $\sigma : T^* \otimes T^* \to S^2 T^*$, then $\pi_1^2(R^{(1)}) = R$ (in fact $H^{0,2}(N) = 0$) but the symbol N which is obviously homogeneous has fibers N_b isomorphic to the symbol described in 4.12 - 3; so $H^{r,2}(N) = 0$ except for $r = 1$. Thus the integrability conditions for R read $\varkappa_1(R^{(1)}) = 0$ and one verifies that this is equivalent to constant sectional curvature for g, as well known.

2. *Scalar equations*. Let $L : C^\infty(B) \to C^\infty(B)$ be a k-th order scalar differential operator with symbol $\sigma_k(L) \in S^k T$ such that $\sigma_k(L)_b \neq 0$ ($\in S^k T_b$), $\forall b \in B$. Then $N = \ker \sigma_k(L) \subset S^k T^*$ is a vector bundle with N_b of codimension 1 in $S^k T_b^*$ for any $b \in B$; by 4.13 - 4 $N_b^{(1)}$ is of codimension n in $S^{k+1} T_b^*$ and $H^{r,2}(N_b) = 0$, $\forall r \geqslant 0$, $\forall b \in B$. It follows (see in the proof of 5.7) that all the $N^{(\ell)}$ are vector bundles ($\ell \geqslant 0$). Let J_k denote the bundle of k-jets of functions on B; $R = \ker p_k(L) \subset J_k$ is a k-th order partial differential equation with symbol N and $\pi_{k-1}^k(R) = J_{k-1}$ so, (it is regular, R_b is of codimension 1 in J_k), since $H^{0,2}(N)$, $H^{1,2}(N), \dots, H^{\ell,2}(N), \dots$ vanish, R is formally integrable, (notice that N is not necessarily homogeneous). Of course, this result is somehow trivial because one already knows it from classical analysis of ordinary partial differential equations.

6. SOME APPLICATIONS TO CLASSICAL FIELD THEORY

6.1. Notations and conventions

Let E be a vector bundle over B equipped with a connection ∇; if $X \in \Gamma(T)$ is a vector field on B, $\nabla_X : \Gamma(E) \to \Gamma(E)$ will denote the composition

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(T^* \otimes E) \xrightarrow{X \otimes \mathrm{Id}} \Gamma(E).$$

If $(\mathcal{O}, x^\lambda)$ denote a chart of B, ∇_λ denotes $\nabla_{\partial/\partial x^\lambda} : \Gamma(E \restriction \mathcal{O}) \to \Gamma(E \restriction \mathcal{O})$. There is a natural connection on the dual vector bundle E^*, again denoted by ∇, characterized by $X\langle s^*, s\rangle = \langle\nabla_X s^*, s\rangle + \langle s^*, \nabla_X s\rangle$ for any $X \in \Gamma(T)$, $s \in \Gamma(E)$ and $s^* \in \Gamma(E^*)$, ($\langle s^*, s\rangle \in C^\infty(B)$). If E' is another vector bundle with a connection again denoted by ∇, there is a natural connection on $E \otimes E'$, (resp. $E \oplus E'$), such that $\nabla_X(s \otimes s') = (\nabla_X s) \otimes s' + s \otimes \nabla_X s'$, (resp. $\nabla_X(s + s') = \nabla_X s + \nabla_X s'$), for any $X \in \Gamma(T)$, $s \in \Gamma(E)$ and $s' \in \Gamma(E')$.

In the following, B will be a (smooth connected pseudo-riemannian manifold with pseudo-metric $g \in \Gamma(S^2 T^*)$; $g^\# : T \to T^*$ denote the bundle isomorphism $X \mapsto g^\# X = g(X, \cdot)$ and $g_\#$ denote its inverse. All tensor bundles $\mathscr{T}$ of B will be equipped with the Levi-Civita connection, ∇. If B has a spin structure with spin bundle $\mathscr{S}$, $\mathscr{S}$ is equipped with the canonical connection ∇ associated with the Levi-Civita connection, the «Clifford product» is denoted by $\gamma \in \Gamma(T^* \otimes \operatorname{End}\mathscr{S})$, $(\gamma(X)\gamma(Y) + \gamma(Y)\gamma(X) = 2g(X, Y))$ and $\gamma_\# : T^* \otimes \mathscr{S} \to \mathscr{S}$ is defined by $\gamma_\#(\omega \otimes \psi) = \gamma(g_\#(\omega))\psi$ for $\omega \in \Gamma(T^*)$ and $\psi \in \Gamma(\mathscr{S})$. $\gamma_\#$ is the symbol of the *Dirac operator* $\nabla\!\!\!/ = \gamma_\# \circ \nabla : \Gamma(\mathscr{S}) \to \Gamma(\mathscr{S})$. In local coordinates, (x^λ), we write $g(X, Y) = g_{\mu\nu} X^\mu Y^\nu$, $g(g_\#(\alpha), g_\#(\beta)) = g^{\mu\nu}\alpha_\mu\beta_\nu$, $\gamma(X) = \gamma_\lambda X^\lambda$, $\gamma(g_\#(\omega)) = \gamma^\lambda\omega_\lambda$ $\nabla\!\!\!/ = \gamma^\lambda\nabla_\lambda$, etc. . . . .

Quite generally we shall always denote by ∇ the connections involved, when no confusion arises, and extend ∇ to vector bundle valued forms as in 2.6. Notice that there is then an ambiguity between $\nabla : \Gamma(\Lambda^p T^* \otimes E) \to \Gamma(\Lambda^{p+1} T^* \otimes E)$ and the connection: $\Gamma(\Lambda^p T^* \otimes E) \to \Gamma(T^* \otimes \Lambda^p T^* \otimes E)$ corresponding to the tensor product of $\Lambda^p T^*$ equipped with the Levi-Civita connection with E, the first is the composition of the second with the exterior product (since the Levi-Civita connection is torsion free). We shall avoid this ambiguity by only using the symbols ∇_X, (or ∇_λ in local coordinates), in the second case.

Finally there is a Hodge operation $* : \Lambda^p T^* \to \Lambda^{n-p} T^*$ associated with g defined by

$$(*\omega)_{\mu_1\ldots\mu_{n-p}} = \sqrt{|\det(g_{\lambda\nu})|}\,\epsilon_{\mu_1\ldots\mu_{n-p}\rho_1\ldots\rho_p}\, g^{\rho_1\sigma_1}\ldots g^{\rho_p\sigma_p}\,\omega_{\sigma_1\ldots\sigma_p}$$

with $\alpha \wedge *\beta = \beta \wedge *\alpha = g(\alpha, \beta)\,\mathrm{vol} \in \Lambda^n T^*$, $\forall \alpha, \beta \in \Lambda^p T^*$. We extend $*$ to vector bundle-valued forms.

In the following we shall analyse some linear equations with homogeneous symbols. The homogeneity will always come from the fact that given $b, b' \in B$ there are linear mappings $T_b \to T_{b'}$, $\mathscr{S}_b \to \mathscr{S}_{b'}$ which transport g_b on $g_{b'}$ and γ_b on $\gamma_{b'}$.

6.2. General Dirac equation

Let $\mathscr{S}$, $\gamma_\#$, ∇ on $\mathscr{S}$ be as above and E be a vector bundle with connection. We have, as above, a natural connection on $\mathscr{S} \otimes E$, again denoted by ∇; define $\nabla\!\!\!/$ on $\mathscr{S} \otimes E$ by $\nabla\!\!\!/ = \gamma_\# \otimes \mathrm{Id} \circ \nabla : \Gamma(\mathscr{S} \otimes E) \to \Gamma(\mathscr{S} \otimes E)$. Let $V : \mathscr{S} \otimes E \to \mathscr{S} \otimes E$

be a vector bundle endomorphism and consider the 1-th order linear equation $R = \ker p_1(\tilde{\nabla} + V)$ on $\mathcal{S} \otimes E$; in local trivialisation $\mathcal{S} \otimes E \restriction \mathcal{C} = \mathcal{C} \times E_0$ and local coordinates, R reads

$$\gamma(x)^{\lambda} \nabla_{\lambda} \psi(x) + V(x)\, \psi(x) = 0 \qquad (R).$$

The symbol $N = \ker(\gamma_{\#} \otimes \mathrm{Id})$ of R is homogeneous and for $b \in B$, N_b is the involutive symbol described in 4.12 - 6. By counting the dimensions one easily sees that $H^{0,2}(N_b) = 0$, so since we have $\pi_0^1(R) = \mathcal{S} \otimes E$, $((\gamma^{\lambda})^2 = g^{\lambda\lambda})$, R is formally integrable ($\varkappa_{\ell} = 0\ \forall \ell \geqslant 0$).

6.3. Rarita-Schwinger equation [6] (Spin 3/2, zero mass)

Let $L : \Gamma(T^* \otimes \mathcal{S}) \to \Gamma(\Lambda^{n-1} T^* \otimes \mathcal{S})$ be the 1-th order operator defined by $L\psi = \underbrace{\gamma \wedge \ldots \wedge \gamma}_{n-3} \wedge \nabla \psi$, for $\psi \in \Gamma(T^* \otimes \mathcal{S})$, and consider the 1-th order linear equation $R = \ker p_1(L)$ on $T^* \otimes \mathcal{S}$. We have $\pi_0^1(R) = T^* \otimes \mathcal{S}$ and the symbol N of R is homogeneous and reduces for each $b \in B$ to the involutive symbol N_b isomorphic with the symbol described in 4.12 - 7. So R is formally integrable if and only if $\pi_1^2(R^{(1)}) = R$ or, which is the same here, if and only if $\varkappa_0(R) = 0$. Applying ∇ to $L\ \psi = 0$ we obtain $\underbrace{\gamma \wedge \ldots \wedge \gamma}_{n-3} \wedge \Omega \wedge \psi = 0$ and since $\nabla \circ L$ factorizes through $j^1 \circ L$, it follows that $\pi_1^2(R^{(1)}) \subset R \cap \ker(\gamma \wedge \ldots \wedge \gamma \wedge \Omega \wedge \pi_0^1)$; in fact there are no other independent equations of order less than one, as can be shown by using coordinates, and the vector bundle homomorphism $\gamma \wedge \ldots \wedge \gamma \wedge \Omega \wedge \pi_0^1 : R \to \Lambda^n T^* \otimes \mathcal{S}$ is essentially $\varkappa_0$ $(H^{0,2}(N) \simeq \Lambda^n T^* \otimes \mathcal{S})$. Thus R is formally integrable if and only if $\underbrace{\gamma \wedge \ldots \wedge \gamma}_{n-3} \wedge \Omega = 0$ (Ω being the curvature of ∇ acting on $\Gamma(\mathcal{S})$) and this turns out to be equivalent to the vanishing of the Ricci tensor of (B, g) as one can see by using «γ-gymnastic». So the formal integrability of R is equivalent to the Einstein equations for empty space $R_{\mu\nu}(g) = 0$ [6].

6.4. Yang-Mills equations as integrability condition [3, 4]

1. *Second order form* [3]. Let E be a vector bundle over B with a connection ∇ and let $L : \Gamma(T^* \otimes E) \to \Gamma(\Lambda^{n-1} T^* \otimes E)$ be the 2-th order differential operator defined by

$$L\alpha = (\nabla * (\nabla \alpha) - (-1)^n (*\Omega) \wedge \alpha, \quad \alpha \in \Gamma(T^* \otimes E),$$

where Ω is the curvature of ∇. The 2-th order linear partial differential equation $R = \ker p_2(L) \subset J_2(T^* \otimes E)$ reads in local coordinates and local trivialisation

$$g^{\lambda\mu}(\nabla_{\lambda}(\nabla_{\mu}\alpha_{\nu} - \nabla_{\nu}\alpha_{\mu}) - F_{\mu\nu}\alpha_{\lambda}) = 0,$$

so the symbol N of R is homogeneous and for $b \in B$, N_b reduces, (apart from a trivial tensorisation), to the involutive Maxwell symbol described in 4.12 - 4. Again, by 5.7, R is formally integrable if and only if $\pi_2^3(R^{(1)}) = R$; furthermore we have here $\pi_1^2(R) = J_1(T^* \otimes E)$ so $\varkappa_0$ is well defined (by 5.6) and formal integrability of R is equivalent to $\varkappa_0 = 0$. Let us describe the vector bundle homomorphism $\varkappa_0 : R \to H^{0,2}(N)$. Notice that

$$\nabla L\, \alpha = \Omega \wedge * \nabla \alpha + (-1)^{n-1}(\nabla * \Omega) \wedge \alpha - (*\Omega) \wedge \nabla \alpha = (-1)^{n-1}(\nabla * \Omega) \wedge \alpha$$

and since $\nabla \circ L$ factorizes through $j^1 \circ L$,
$\pi_0^3(R^{(1)}) \subset \ker(\alpha \mapsto (\nabla * \Omega) \wedge \alpha)$ and therefore we have
$\pi_2^3(R^{(1)}) \subset R \cap \ker(\nabla * \Omega) \wedge \pi_0^2)$; it can be checked that the last inclusion is in fact an equality, i.e. there are no other independent equation in $\pi_2^3(R^{(1)})$. In fact, we have $H^{0,2}(N) \simeq E \simeq \Lambda^n T^* \otimes E$, (via the $*$-operation), and $\varkappa_0 : R \to H^{0,2}(N)$ is described by $(\nabla * \Omega) \wedge \pi_0^2 : R \to \Lambda^n T^* \otimes E$. Thus, R is formally integrable if and only if the connection ∇ satisfies the Yang-Mills equation without source $\nabla * \Omega = 0$.

2. *First order form* [3, 4]. Let E, ∇, Ω be as above and let
$\Sigma \in \Gamma(\Lambda^{n-2} T^* \otimes \mathrm{End}(E))$ be a End (E)-valued $(n-2)$-form (End $(E) = E \otimes E^*$). Consider the first order differential operator L from $(T^* \otimes E) \oplus (\Lambda^{n-2} T^* \otimes E)$ in $(\Lambda^2 T^* \otimes E) \oplus (\Lambda^{n-1} T^* \otimes E)$ defined by

$$L(\alpha \oplus \beta) = (\nabla \alpha - \overset{-1}{*} \beta) \oplus (\nabla \beta - (-1)^n \Sigma \wedge \alpha),$$

for $\alpha \in \Gamma(T^* \otimes E), \beta \in \Gamma(\Lambda^{n-2} T^* \otimes E)$.

The symbol $\tilde{N}$ of the first order linear partial differential equation $\tilde{R} = \ker p_1(L)$ is clearly homogeneous and its fibre $\tilde{N}_b$, $b \in B$, reduces to the tensor product of the direct sum $N_{[1]} \oplus N_{[n-2]}$ of exterior differential symbols described in 4.12 - 2 with E_b; so $\tilde{N}$ is involutive and $\tilde{R}$ is formally integrable if and only if $\pi_1^2(\tilde{R}^{(1)}) = \tilde{R}$. $\pi_0^1(\tilde{R}) = (T^* \otimes E) \oplus (\Lambda^{n-2} T^* \otimes E)$ so $\varkappa_0$ is well defined and $\tilde{R}$ is formally integrable iff. $\varkappa_0 = 0$. We have:

$$\nabla(\nabla \beta - (-1)^n \Sigma \wedge \alpha) = \Omega \wedge \beta - \Sigma \wedge \nabla \alpha - (-1)^n (\nabla \Sigma) \wedge \alpha$$

which reduces on $\tilde{R}$ (by $\nabla \alpha - \overset{-1}{*} \beta = 0$) to $(\Omega - \overset{-1}{*} \Sigma) \wedge \beta - (-1)^n (\nabla \Sigma) \wedge \alpha$; since, by construction, this factorizes through $j^1 \circ L(\alpha \oplus \beta)$, we have

$$\pi_0^2(\tilde{R}^{(1)}) \subset \ker(\alpha \oplus \beta \mapsto (\Omega - \overset{-1}{*} \Sigma) \wedge \beta - (-1)^n (\nabla \Sigma) \wedge \alpha)$$

and in fact the composition of

$$\alpha \oplus \beta \mapsto (\Omega - \overset{-1}{*} \Sigma) \wedge \beta - (-1)^n (\nabla \Sigma) \wedge \alpha \quad \text{with} \quad \pi_0^1 \restriction \tilde{R}$$

is a vector bundle homomorphism of $\tilde{R}$ into $\Lambda^n T^* \otimes E$ which is essentially

$\varkappa_0 : \tilde{R} \to H^{0,2}(\mathcal{N})$. Thus $\tilde{R}$ is formally integrable if and only if $\Sigma = *\,\Omega$ and $\nabla\Sigma = 0$ (i.e. $\nabla *\Omega = 0$) which is the first order form of Yang-Mills equation:

$$*\,(\nabla^2) - \Sigma = 0 \quad \text{and} \quad \nabla\Sigma = 0.$$

6.5. Einstein Equations as integrability condition [3, 4, 5, 6]

1. *Second order form* [3, 5]. Let $h \in \Gamma(S^2T^*)$, then for $t \in \mathbb{R}$ sufficiently small, $g + t\,h = g_t$ is a one-parameter family of pseudo-metric on B with $g = g_0$; the Ricci tensor of $g + t\,h$, $\mathrm{Ric}\,(g + t\,h) \in \Gamma(S^2T^*)$, is well defined and differentiable in t in a neighbourhood of $t = 0$. Consider $h \mapsto L\,h = \dfrac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Ric}\,(g + t\,h)\big|_{t=0}$. L is a second order differential operator from S^2T^* in S^2T^*. ($L \in \mathscr{D}_2(S^2T^*, S^2T^*)$), which reads in local coordinates:

$$(L\,h)_{\mu\nu} = g^{\lambda\rho}(\nabla_\lambda\nabla_\mu h_{\nu\rho} + \nabla_\lambda\nabla_\nu h_{\mu\rho} - \nabla_\lambda\nabla_\rho h_{\mu\nu} - \nabla_\mu\nabla_\nu h_{\lambda\rho}).$$

It follows that the second order linear partial differential equation $R = \ker p_2(L)$ on S^2T^* has a homogeneous symbol $\mathcal{N}$ with typical fibre isomorphic with the involutive symbol of example 4.12 - 5. Furthermore, we have $\pi_1^2(R) = J_1(S^2T^*)$, so $\varkappa_0 : R \to H^{0,2}(\mathcal{N})$ is well define and vanish whenever $\pi_2^3(R^{(1)}) = R$ i.e. whenever R is formally integrable. By using the identity $\nabla_\lambda R^\lambda{}_{\mu\nu\rho} = \nabla_\mu R_{\nu\rho} - \nabla_\nu R_{\mu\rho}$, one obtains

$$g^{\tau\mu}\left(\nabla_\tau(L\,h)_{\mu\nu} - \frac{1}{2}\,\nabla_\nu(L\,h)_{\mu\tau}\right) =$$

$$g^{\tau\mu}g^{\lambda\rho}\{R_{\tau\nu}\nabla_\lambda(2h_{\mu\rho} - g_{\mu\rho}g^{\alpha\beta}h_{\alpha\beta}) + (\nabla_\lambda R_{\tau\nu} - \nabla_\nu R_{\lambda\tau})\,(h_{\mu\rho} - g_{\mu\rho}g^{\alpha\beta}h_{\alpha\beta})\}.$$

Since the first side factorizes through $j^1 \circ L(h)$, it is necessary that $R_{\mu\nu} = 0$ in order that $\pi_2^3(R^{(1)}) = R$, it can be shown that this is also sufficient. (this follows for instance from integrability of $\mathrm{Ric}\,(g) = 0$ since then R is the linearisation of $\mathrm{Ric}\,(g) = 0$). Thus the linear equation R is formally integrable if and only if $\mathrm{Ric}\,(g) = 0$, i.e. if g satisfies the free Einstein equations. Notice that $\pi_2^3(R^{(1)}) = R$ iff. we have $\pi_1^3(R^{(1)}) = J_1(S^2T^*)$ and, in fact, $\varkappa_0 : R \to H^{0,2}(\mathcal{N})$ factorize through $\pi_1^2 : R \to J_1(S^2T^*)$.

2. *First order form*. In this paragraph, we let g be, as above, a pseudometric on B but ∇ denotes now *an arbitrary* torsion-free linear connection on B which is not necessarily the Levi-Civita connection, (i.e. *we do not assume* that we have $\nabla g = 0$). (Consider the first order linear partial differential equation, $\tilde{R}$, on $S^2T^* \oplus (T \otimes S^2T^*)$ which reads in local coordinates for $h + \Delta \in S^2T^* \oplus (T \otimes S^2T^*)$:

$$\nabla_\lambda h_{\mu\nu} - g_{\mu\rho}\Delta^\rho{}_{\nu\lambda} - g_{\nu\rho}\Delta^\rho{}_{\mu\lambda} = 0 \qquad (\tilde{R}_1)$$

$$\nabla_\lambda \Delta^\lambda{}_{\mu\nu} - \nabla_\nu \Delta^\lambda{}_{\mu\lambda} = 0 \qquad (\tilde{R}_2).$$

Notice first that we have: $\pi_0^1(\tilde{R}) = S^2T^* \oplus (T \otimes S^2T^*)$. By $(\tilde{R}_2)$ we have $\nabla_\nu \Delta^\lambda{}_{\mu\lambda} - \nabla_\mu \Delta^\lambda{}_{\nu\lambda} = 0$, so that, by taking covariant derivatives of $(\tilde{R}_1)$ we obtain $\nabla_\mu(g^{\lambda\rho}\nabla_\nu h_{\lambda\rho}) = \nabla_\nu(g^{\lambda\rho}\nabla_\mu h_{\lambda\rho})$ which implies

$$(\nabla_\mu g^{\lambda\rho}\delta^\sigma_\nu - \nabla_\nu g^{\lambda\rho}\delta^\sigma_\mu)(g_{\lambda\tau}\Delta^\tau{}_{\rho\sigma} + g_{\rho\tau}\Delta^\tau{}_{\lambda\sigma}) \ + g^{\lambda\rho}(R^\sigma{}_{\lambda,\mu\nu}\delta^\tau_\rho + R^\sigma{}_{\rho,\mu\nu}\delta^\tau_\lambda)h_{\sigma\tau} = 0,$$

where we used $(\tilde{R}_1)$ and $R^\alpha{}_{\beta,\gamma\delta}$ denoteds the curvature tensor of ∇. It follows that $\pi_0^2(\tilde{R}^{(1)}) = S^2T^* \oplus (T \otimes S^2T^*)$ implies $\nabla g = 0$, i.e. implies that ∇ is the Levi-Civita connection. Assume now that ∇ is the Levi-Civita connection, then the system $\tilde{R}$ is just the first order form of the system R of the last paragraph 6.5 - 1, and we saw there that $\pi_1^3(R^{(1)}) = J_1(S^2T^*)$ is equivalent to $\mathrm{Ric}\,(g) = 0$ and that this is equivalent to formal integrability R; it follows that here, $\pi_0^3(\tilde{R}^{(2)}) = S^2T^* \oplus (T \otimes S^2T^*)$ implies $\mathrm{Ric}\,(g) = 0$ and, finally, formal integrability.

Summarizing, we saw that for $\tilde{R}$, $\pi_0^3(\tilde{R}^{(2)}) = S^2T^* \oplus (T \otimes S^2T^*)$ implies the first order form form of Einstein equations $\nabla g = 0$, $R^\lambda{}_{\mu,\lambda\nu}(\nabla) = 0$ for (g, ∇) which in turn implies formal integrability of $\tilde{R}$, and thus, these 3 statements for $\tilde{R}$ are equivalent.

Notice that, if (g, ∇) satisfies the first order form of Einstein equations, then $\tilde{R}$ is the linearization of these Einstein equations around (g, ∇), i.e. the first order in ϵ-expansion of these equations for $(g + \epsilon h, \nabla + \epsilon\Delta)$.

3. Remark

Notice that, in view of 6.3, the Einstein equations are also the formal integrability conditions of the first order system described there [6]. However, in 6.5 - 2, the symbol of $\tilde{R}$ in a «constant symbol» (it is invariant by diffeomorphisms) and the «coefficients» of $\tilde{R}$ do only depend, in an affine way, on (∇, g) which are the variables of the first order Einstein equations $\nabla g = 0$, $R^\lambda{}_{\mu,\lambda\nu}(\nabla) = 0$ i.e. the formal integrability of $\tilde{R}$ implies both $\mathrm{Ric}\,(g) = 0$ and the correct «contact condition» $\nabla g = 0$. Let us now come back to the structures described in 6.4 - 2 and 6.5 - 2.

6.6. PROPOSITION. *Let $\tilde{E}$ be a vector bundle on B with a connection $\tilde{\nabla}$ and let $\sigma : \Gamma(T^* \otimes \tilde{E}) \to \Gamma(F)$ be a bundle homomorphism with kernel $\tilde{N}$. Suppose that the first order equation $\tilde{R} = \ker p_1(\sigma \circ \tilde{\nabla})$ is formally integrables whenever $\pi_0^{\ell+1}(\tilde{R}^{(\ell)}) = \tilde{E}$, for some $\ell \geqslant 1$.*

a) *Suppose that each $b \in B$ has a neighbourhood $\mathcal{O} \subset B$ such that $\tilde{E} \restriction \mathcal{O}$ admits a flat connection of the form $\tilde{\nabla} + \omega$ with $\omega \in \Gamma_{\mathcal{O}}(\tilde{N} \otimes \tilde{E}^*)$; then $\tilde{R}$ is formally* *integrable.*

b) *Suppose that B is analytic, $\tilde{E}$ is an analytic and that $\tilde{R}$ is a formally integrable analytic equation; then each $b \in B$ has a neighbourhood $C \subset B$ such that $\tilde{E} \restriction C$ admits a flat connection of the form $\tilde{\nabla} + \omega$ with $\omega \in \Gamma_C(\tilde{N} \otimes \tilde{E}^*)$.*

In other words if $\tilde{R}$ is as above and is analytic, then, formal integrability of $\tilde{R}$ is locally equivalent to the existence of flat connections in the affine subbundle $\tilde{\nabla} + \tilde{N} \otimes \tilde{E}^*$ of the affine bundle $\tilde{\nabla} + T^* \otimes \tilde{E} \otimes \tilde{E}^*$ of all connections on $\tilde{E}$.

a) is straightforward, if $\zeta \in \tilde{E}$, any horizontal section s for $\tilde{\nabla} + \omega$ passing through ζ is such that $j_b^{s+1}(s) \in \tilde{R}_b^{(s)}$ since $\sigma(\tilde{\nabla} + \omega)s = \sigma \circ \tilde{\nabla} s = 0$, so $\pi_0^{s+1}(\tilde{R}^{(s)}) = \tilde{E}$ and $\tilde{R}$ is formally integrable.

Let us prove b). If $\tilde{R}$ is formally integrable and analytic, it follows from the theorem 3.4 that, for any $b \in B$ and $\zeta \in \tilde{E}_b$ there is a local solution s_ζ of $\tilde{R}$ with $s_\zeta(b) = \zeta$. So let $e_1, \dots, e_r$ be a basis of $\tilde{E}_b$ and let $s_1, \dots, s_r$ be local solutions of R with $s_1(b) = e_1, \dots, s_r(b) = e_r$ then $s_1(b'), \dots, s_r(b')$ are basis $\tilde{E}_{b'}$ in a neighbourhood of b and there is a unique connection in this neighbourhood for which they are horizontal; this connection is of the form $\tilde{\nabla} + \omega$ with $\omega(b') \in \tilde{N} \otimes \tilde{E}^*$ since by assumption $\sigma \tilde{\nabla} s_k = 0$; $\tilde{\nabla} + \omega$ is flat by construction.

6.7. Remark

The above proof of b) shows that, when $\tilde{R}$ is a formally integrable analytic equation as above, then there are in general several flat local connections as above and the theorem 3.4 shows that we may choose these connections to be analytic.

6.8. Yang-Mills and Einstein Equations as zero curvature conditions

We now show that the systems $\tilde{R}$ of 6.4 - 2 and 6.5 - 2 satisfy the assumption of 6.6.

1. *Yang-Mills*. Let us use the notation of 6.4 - 2. Set
$\tilde{E} = (T^* \otimes E) \oplus (\Lambda^{n-2} T^* \otimes E)$ and let

$\sigma : T^* \otimes \tilde{E} = (T^* \otimes T^* \otimes E) \oplus (T^* \otimes \Lambda^{n-2} T^* \otimes E) \to (\Lambda^2 T^* \otimes E) \oplus (\Lambda^{n-1} T^* \otimes E)$

correspond to the exterior product ($T^* \otimes T^* \to \Lambda^2 T^*$ and $T^* \otimes \Lambda^{n-2} T^* \to \Lambda^{n-1} T^*$). Define the connection $\tilde{\nabla}$ on $\tilde{E}$ by:

$$\tilde{\nabla}_X(\alpha \oplus \beta) = \left(\nabla_X \alpha - \frac{1}{2}\, i(X) \ast^1 \beta\right) \oplus (\nabla_X \beta - \Sigma\, i(X)\alpha);$$

where $X \in \Gamma(T)$ and $i(X)$ is defined on $\Gamma(\Lambda^{\cdot} T^*)$ to be the unique $C^\infty(B)$-linear antiderivation such that $i(X)\, C^\infty(B)1 = 0$ and
$i(X)\omega = \langle X, \omega\rangle \in C^\infty(B) = \Gamma(\Lambda^0 T^*)$ for any 1-form $\omega \in \Gamma(T^*)$; $i(X)$ is extended to $\Gamma(\Lambda T^* \otimes E)$ by $i(X)(\omega \otimes s) = (i(X)\omega) \otimes s$, $\omega \in \Gamma(\Lambda^{\cdot} T^*)\, s \in \Gamma(E)$. With these notations we have $L = \sigma \circ \tilde{\nabla}$ so the equation $\tilde{R}$ of 6.4 - 2 is of the form $\tilde{R} = \ker p_1(\sigma \circ \tilde{\nabla})$ and we know from the discussion of 6.4 - 2 that $\tilde{R}$ is formally integrable whenever $\pi_0^2(\tilde{R}^{(1)}) = \tilde{E}$. Since we know that this formal integrability is equivalent to the first order form of Yang-Mills equations, we conclude, by 6.6, that if E is an analytic vector bundle and if ∇, Σ are analytic, then, ∇, Σ satisfy the first order Yang-Mills equations $*\nabla^2 = \Sigma$ and $\nabla\Sigma = 0$ if and only if there are local sections ω of $\tilde{N} \otimes \tilde{E}$ such that $(\tilde{\nabla} + \omega)^2 = 0$, (i.e. $\tilde{\nabla} + \omega$ is flat). Notice that the mapping $(\nabla, \Sigma) \mapsto \tilde{\nabla}$ defined above is an injective homomorphism of affine bundles $A(E) \oplus \Lambda^{n-2} T^* \otimes E \otimes E^* \to A(\tilde{E})$, where $A(E)$ is the affine bundle the sections of which are the connections on E.

2. *Einstein*. Let us use the notations of 6.5 - 2. Set $\tilde{E} = S^2 T^* \oplus (T \otimes S^2 T^*)$, $\tilde{\nabla}_X(h \oplus \Delta) = (\nabla_X h - k(X)\Delta) \oplus \nabla_X \Delta$, where

$$X \in \Gamma(T),\ (k(X)\Delta)_{\mu\nu} = g_{\mu\rho}\, \Delta^{\rho}{}_{\nu\lambda} X^{\lambda} + g_{\nu\rho}\, \Delta^{\rho}{}_{\mu\lambda} X^{\lambda}.$$

$\tilde{\nabla}$ is a connection on $\tilde{E}$. Let σ be the symbol of the right hand side of $(\tilde{R})$, $\sigma : T^* \otimes \{S^2 T^* \oplus (T \otimes S^2 T^*)\} \to (T^* \otimes S^2 T^*) \oplus S^2 T^*$. Then $(\tilde{R})$ reads $\sigma \circ \tilde{\nabla}\,(h \oplus \Delta) = 0$ so we have $\tilde{R} = \ker p_1(\sigma \circ \tilde{\nabla})$ and we know from the discussion in 6.5 - 2 that $\tilde{R}$ is formally integrable whenever $\pi_0^3(\tilde{R}^{(2)}) = \tilde{E}$ and that this is equivalent for (g, ∇) to the first order form of Einstein equations $\nabla g = 0$ and $R^{\lambda}{}_{\mu,\lambda\nu}(\nabla) = 0$. Thus, again by 6.6, if B, g and ∇ are analytic, the (g, ∇) satisfy the first order form of Einstein equations iff. there are local sections ω of $\tilde{N} \otimes \tilde{E}^*$ such that $\tilde{\nabla} + \omega$ is flat ($(\tilde{\nabla} + \omega)^2 = 0$).

Let $A_0(T)$ be the affine subbundle of $A(T)$ the sections of which are the torsion-free (i.e. symmetric) connection and $S^2_{\epsilon} T^*$ be the cone of elements of $S^2 T^*$ which are non-degenerated of signature ϵ. Then $(\nabla, g) \mapsto \tilde{\nabla}$ is the restriction to $A_0(T) \oplus S^2_{\epsilon} T^*$ of an injective affine homomorphism from $A_0(T) \oplus S^2 T^*$ in $A(\tilde{E})$.

7. CONCLUSION AND OUTLOOKS

We have shown that the Yang-Mills and the Einstein equations are the integrability conditions of linear systems and that, even more, that can be read as zero curvatures for connections of some types on appropriate vector bundles. This is to be compared with what happens [2] for the so-called «completely integrable» systems of partial differential equations in dimension 2. In dimension 2, however, the pure Yang-Mills and Einstein equations are essentially trivial from the local point of view. In dimension n greater than 2, one loses the relation between zero-curvature and conserved quantity (because, then, we have $\pi_1(S^{n-1}) = 0$). There is nevertheless a way to produce, in principle, infinite sets of conserved quantities which comes from the following property of the linear systems that we used. When the linear system is integrable, i.e. when Yang-Mills or Einstein equations are satisfied, one can construct with two solutions φ_1, φ_2 of the linear system, a closed $(n-1)$-form $\omega(\varphi_1, \varphi_2)$, (i.e. by *, a conserved current $J^\mu(\varphi_1, \varphi_2)$), which is bilinear and local in φ_1, φ_2, and such that its restriction to a local Cauchy surface does only depend on φ_1 and φ_2 through their local Cauchy data on the surface: so by fixing sets of independent Cauchy data for the linear system one obtains a (infinite) set of conserved quantities that only depend on the coefficients of the linear equation, i.e. on Yang-Mills or Einstein fields. Considering Yang-Mills or Einstein equations as dynamical systems one may try to compute the Poisson bracket ot these conserved quantities and to exhibit the corresponding Lie algebra. Practically, however, this is very difficult because all these systems are systems with constraints, so one must use the whole machinery to deal with such systems both at the level of the linear systems and at the level of the (non--linear) Yang-Mills or Einstein systems. Work in this direction is currently in progress.

There is another aspect of the interpretation of Yang-Mills and Einstein equations in terms of integrability condition, which was pointed out in a previous work [3] and was then used to generalise to the coupled Yang-Mills charged field equations and to the coupled gravitation-matter field equations the above discussion: it is the interplay between these integrability problems and the invariance by «infinite groups» (or pseudogroups), namely gauge invariance and invariance by diffeomorphism. For instance, the free Einstein equations, $\mathrm{Ric}(g) = 0$, are the Euler-Lagrange equations corresponding to the functional $g \mapsto S = \int g^{\mu\nu} R_{\mu\nu}(g)\,\mathrm{vol}$, this functional is invariant by diffeomorphism which leads to the identity $\nabla_\lambda g^{\lambda\mu}\left(R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} g^{\alpha\beta} R_{\alpha\beta}\right) = 0$ via the second Noether theorem [19]. Taking the derivative of this identity written for $g + t\,h$ at $t = 0$, one obtains an identity connecting first derivatives of L (i.e. $j^1 \circ L$) with the Ricci tensor [3, 5] and it is this very identity which implies that $\pi_2^3(R^{(1)}) = R$ is equivalent to $\mathrm{Ric}(g) = 0$, where $R = \ker p_2(L)$, $R^{(1)} = \ker p_3(j^1 \circ L)$. Similar considerations apply to Einstein-matter field equations which also come from actions invariant by diffeomorphisms. For Yang-Mills equations and coupled Yang-Mills-charged field equations, the role of diffeomorphisms is played by the gauge transformations. Notice that gauge transformations are diffeomorphisms of special kind of the appropriate principal bundle and it is known that, there, the Yang-Mills current may be interpreted as a part of Ricci tensor [20, 21].

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References

[1] *Integrable quantum field theories*, Lecture Notes in Physics 151, J. Hietarinta and C. Montonen, Eds. Springer-Verlag (1982).

[2] L. FADDEEV, *Integrable models in 1 + 1 dimensional quantum field theory*, Preprint S. Ph. T./82/72, to appear in the proceedings of Les Houches 1982.

[3] M. DUBOIS-VIOLETTE, *Einstein equations, Yang-Mills equations and classical field theory as compatibility conditions of linear partial differential operators*, Phys. Lett. **119B** (1982) 157 - 161.

[4] M. DUBOIS-VIOLETTE, *Remarks on the local structure of Yang-Mills and Einstein equations*, Phys. Lett. **131B** (1983) 323 - 326.

[5] J. GASQUI, *Sur la résolubilité locale des équations d'Einstein*, Compositio Mathematica **47** (1982) 43 - 69.

[6] B. JULIA, *Système linéaire associé aux équations d'Einstein*, C.R. Acad. Sci. Paris **295** Série II (1982) 113 - 116.

[7] *Invariant wave equations*, Lecture Notes in Physics 73, G. Velo and A.S. Wightman, Eds. Springer-Verlag (1978).

[8] D.C. SPENCER, *Overdetermined systems of linear partial differential equations*, Bull. A.M.S. **75** (1969) 179 - 239.

[9] D.C. SPENCER, (a) *Deformation of structures on manifolds defined by transitive, continuous pseudogroups*,
I, II, Ann. of Math. (2) **76** (1962) 306 - 445;
III, Ann. of Math. (2) **81** (1965) 389 - 450.
(b) *A formal exterior differentiation associated with pseudogroups*, Scripta Math. **26** (1961) 101 - 106.

[10] D.G. QUILLEN, *Formal properties of overdetermined systems of linear partial differential equations*, Thesis, Harvard University, Cambridge, Mass., 1964.

[11] H. GOLDSCHMIDT, *Existence theorems for analytic linear partial differential equations*, Ann. of Math. (2) **86** (1967), 247 - 270.

[12] H. GOLDSCHMIDT, *Integrability criteria for systems of non-linear partial differential equations*, J. Differential Geometry **1** (1967), 269 - 307.

[13] B. MALGRANGE, *Cohomomogie de Spencer (d'après Quillen)*, Secrétariat Mathématique d'Orsay, 1966.

[14] B. MALGRANGE, *Théorie analytique des équations différentielles*, Séminaire Bourbaki n° 329, (1967).

[15] V. GUILLEMIN, *Some algebraic results concerning the characteristics of overdetermined partial differential euqations*, Amer. J. Math. **90** (1968) 270 - 284.

[16] C. EHRESMANN, *Les prolongements d'une variété différentiable*, C.R. Acad. Sci., Paris, **233** (1951) 598, 777, 1081; **234** (1952) 1028, 1424.

[17] V. GUILLEMIN, S. STERNBERG, *An algebraic model for transitive differential geometry*, Bull. A.M.S. **70** (1964) 16 - 47 (see the letter by J.P. Serre at the end of the paper).

[18] J. GASQUI, *Sur l'existence locale de certaines métriques riemanniennes plates*. Duke Math. J. **46** (1979) 109 - 118.

[19] A. TRAUTMAN, *Foundations and current problems of general relativity*, in «Lectures on general relativity», A. Trautman, F.A.E. Pirani, H. Bondi, Eds. Brandeis Summer Institute 1964.

[20] B. de WITT, *Dynamical theories of groups and fields*, Gordon and Breach, 1965.

[21] R. KERNER, *Generalization of the Kaluza-Klein theory for an arbitrary non-abelian gauge group*, Ann. Inst. H. Poincaré **9** (1968), 143 - 152.

See also in M. DUBOIS-VIOLETTE, *Equations de Yang-Mills, modèles σ à deux dimensions et généralisations*, in «Mathématique et Physique», Séminaire E.N.S., L. Boutet de Monvel, A. Douady, J. Verdier, Eds. Progress in Math. 37, Birkhaüser 1983.

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